Body motion in a resistive medium: an exactly solvable model

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We introduce and solve in closed form, using momentum and kinetic energy balance, a simplified microscopic model of a body propagating in a one dimensional resistive medium. For a whole family of collisions with varying degree of inelasticities, we find that the effective resistive force on the moving body is opposite to and proportional to the square of the body's velocity.

Keywords: Air drag; collisions

Se plantea y resuelve en forma exacta, usando balance del momentum y energía cinética, un modelo microscópico simplificado de un cuerpo propagándose en un medio resistivo unidimensional Para toda una familia de colisiones con diferente grado de inelasticidad, encontramos que la fuerza efectiva sobre el cuerpo es opuesta y proporcional al cuadrado de la velocidad del cuerpo.

Descriptores: roce viscoso; colisiones

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1. Introduction

When students of elementary physics courses are introduced to the topic of macroscopic bodies moving through a resistive medium [1], they are usually told by the instructor that, according to whether the body has a small or large cross sectional area, or whether the body is moving at a low or high speed, the effective resistive force is either proportional to the body's speed (Stoke's law) [2] or to the square of the body's speed (Newton's law) [3]. There is usually no much discussion about the origin of these 'force laws' and at most, the instructor will tell the students that these 'laws' are based on experimental observations [4]. The topic is admittedly a difficult one, but there are however, simple models that show how these 'force laws' arise from considerations of momentum and energy exchange between the moving body and the medium surrounding it [5]. The mechanical model presented in this work is extremely simple but has the advantage of being exactly solvable and give clues as to the relative importance of dimensionality and the character of the interactions between the body and the medium.

2. The model

Let us consider a (macroscopic) body of mass M propagating in a one-dimensional resistive medium modelled by a set of identical point masses $m < M$, initially at rest. We assume all interactions to be of the 'billiard-ball' type. The collisions among the medium particles are assumed to be completely elastic, while the collisions between the body and a medium particle, is characterized by a 'restitution' coefficient ϵ that determines the degree of elasticity of the collision. The (constant) restitution coefficient ϵ for the body-particle collision is defined as the ratio of the magnitudes of the velocities after the collision ('a') to the one before the collision ('b'):

$$\epsilon = \frac{|V_{\text{body}} - V_{\text{particle}}|_a}{|V_{\text{body}} - V_{\text{particle}}|_b} \qquad (0 \leq \epsilon \leq 1). \tag{1}$$

Thus, when $\epsilon = 1$ we have a completely elastic collision, where the relative magnitude of the body-particle velocity is conserved, while at $\epsilon = 0$, we have the case of a completely inelastic collision, where the body and the particle remain 'glued' to each other after colliding. Let V_0 be the initial speed of the body, and V_1, v the speeds of the body and particle respectively, after the first collision. From conservation of momentum, we have

$$MV_0 = MV_1 + mv. \tag{2}$$

From Eqs.(2) and (1) we obtain

$$V_1 = \left(\frac{M - \epsilon m}{M + m}\right) V_0, \tag{3}$$

$$v = \left(\frac{M + \epsilon M}{M + m}\right) V_0. \tag{4}$$

After this first collision event, the body will continue to move in the original direction (because $M > m > \epsilon m$) with speed V_1 while the first medium particle will recoil with speed $v > V_1$. After some time, the first medium particle will collide with the second medium particle (since the system is one-dimensional, all medium particles can be labelled unambiguously): since both particles have identical mass, the first particle will come to rest, while its momentum will get completely transferred to the second particle (remember that we are assuming fully elastic collisions among the medium particles). The second particle in turn, will transfer all of its momentum to the third particle, and so on. The body, on the other hand, will collide again with the first particle (now at rest). After this second collision, the body will emerge with speed

$$V_2 = \left(\frac{M - \epsilon m}{M + m}\right) V_1 = \left(\frac{M - \epsilon m}{M + m}\right)^2 V_0. \tag{5}$$

The new momentum acquired by the first medium particle will be again carried away to the end of the system *without any backscattering*. After n body-particle collision events, the speed of the body will be

$$V_n = \left(\frac{M - \epsilon m}{M + m}\right)^n V_0. \tag{6}$$

If we denote by x the distance travelled by the body between its first and n-th collision, we have, $n = \rho x$, where ρ is the linear density of medium particles per unit length. As in hydrodynamics, we assume that an element of length Δx while 'small', will contain a large number of medium particles. Re-expressing (6) in terms of x, we have

$$V(x) = \left(\frac{1 - \epsilon r}{1 + r}\right)^{\rho x} V_0, \tag{7}$$

where $r \equiv m/M$. We see that the speed of the body decreases exponentially with distance. This can be seen by rewriting (7) as $V(x) = \exp(-\beta x)$ with $\beta \equiv \rho \log\left[(1 + r)/(1 - \epsilon r)\right]$ acting as the effective spatial speed decay rate.

The acceleration of the body can be computed from $a(x) = dV(x)/dt = V(x)dV(x)/dx$. Since $dV(x)/dx = \rho V(x) \log\left[(1 - \epsilon r)/(1 + r)\right]$, we have, $a(x) = \rho \log\left[(1 - \epsilon r)/(1 + r)\right] V^2(x)$. The average effective force on the body as it travels through the resistive medium will then be

$$F = M\,a = -\gamma\, V^2, \tag{8}$$

where

$$\gamma = \frac{\rho m}{r} \log\left(\frac{1 + r}{1 - \epsilon r}\right) \tag{9}$$

is the resistive coefficient.

The velocity of the body as a function of time can be obtained upon integration of Eq. (8):

$$V(t) = \frac{V_0}{1 + b\, V_0 t}, \tag{10}$$

where

$$b = \rho \log\left(\frac{1 + r}{1 - \epsilon r}\right). \tag{11}$$

The distance travelled by the body as a function of time can, in turn be obtained from integration of Eq.(10). Taking $x(0) = 0$ as the initial condition, one obtains

$$x(t) = \frac{\log(\,1 + b\, V_0 t\,)}{b}. \tag{12}$$

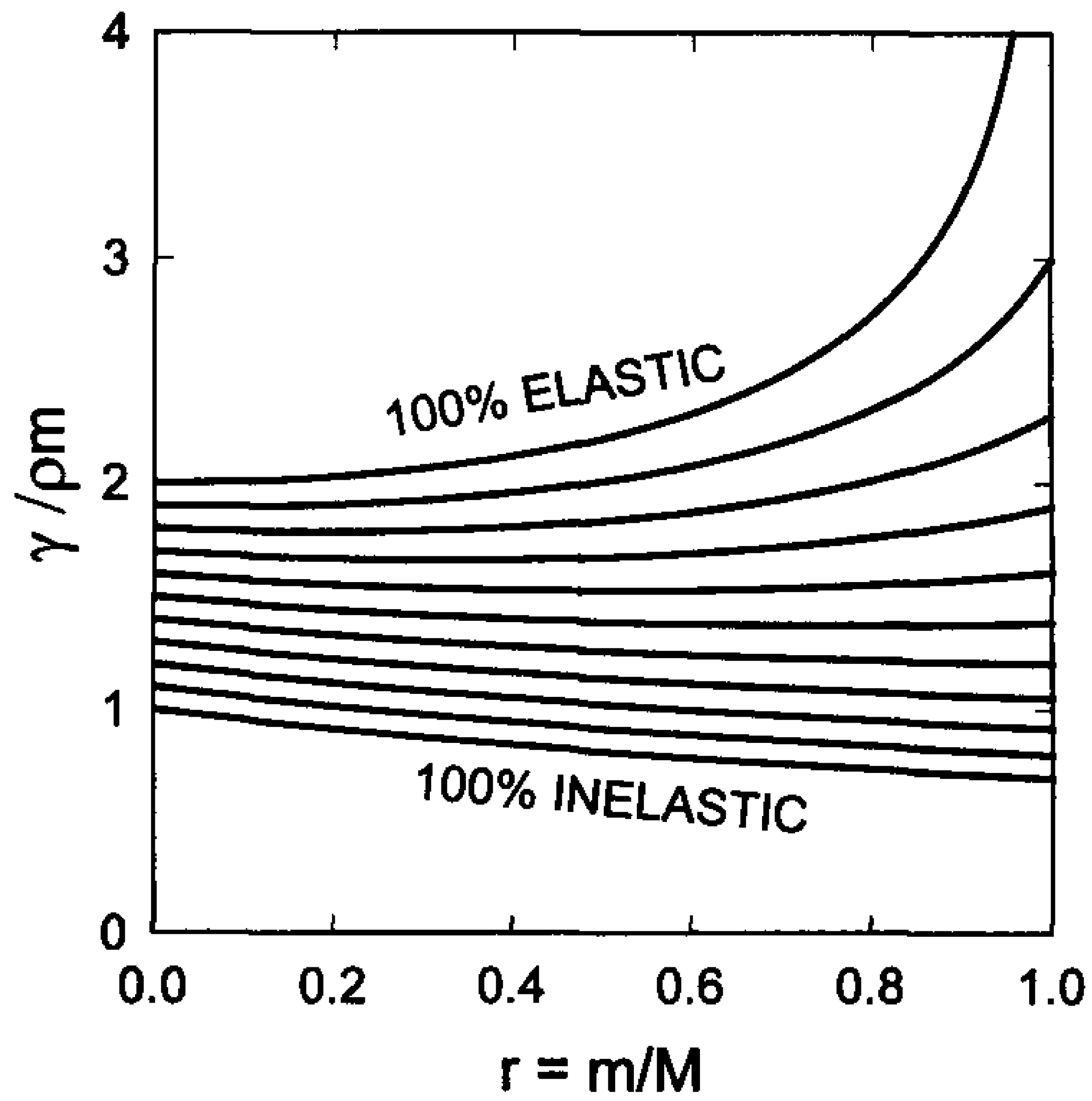

FIGURE 1. Resistive coefficient in terms of the particle/body mass ratio, for several restitution parameter values. From top to bottom: $\epsilon = 1$ (completely elastic), 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1 and 0 (completely inelastic).

The observable that quantifies the energy loss of a projectile as it propagates through a resistive medium is known as the 'stopping power', defined by

$$S(E) = \frac{1}{\rho}\frac{dE(x)}{dx}, \tag{13}$$

where E and ρ are the energy of the projectile and the number density of medium particles. For our model, $E(x) = (1/2)MV(x)^2$, where $V(x) = V_0 \exp(-\beta x)$ with $\beta = \rho \log\left[(1 + r)/(1 - \epsilon r)\right]$. Therefore,

$$S(E) = 2\, \log\left(\frac{1 + r}{1 - \epsilon r}\right) E, \tag{14}$$

i.e., the energy loss per unit length is directly proportional to the body's energy.

3. Results and discussion

For a whole family of inelastic collisions characterized by the restitution coefficient ϵ with $0 \leq \epsilon \leq 1$, which includes the cases of a completely inelastic collision ($\epsilon = 0$) and the completely elastic one ($\epsilon = 1$), we have obtained that the resistive force on the body is always opposite and proportional to the square of the body's speed: $F = -\gamma V^2$. The resistive coefficient γ is shown in Fig. 1 as a function of the mass ratio r and for several degrees of elasticity. We note that $\gamma/\rho m$ is most sensitive to r for the completely elastic case and actually diverges at $r = 1$. In this case, both the body and the medium particle have the same mass, which makes the body come to a dead stop right after its first collision. For nonzero inelasticity, γ becomes less sensitive to r and, around $r = 0.6$, it seems to become quite insensitive. Let us find the conditions under which $\gamma/\rho m$ is most insensitive to r. First, we define

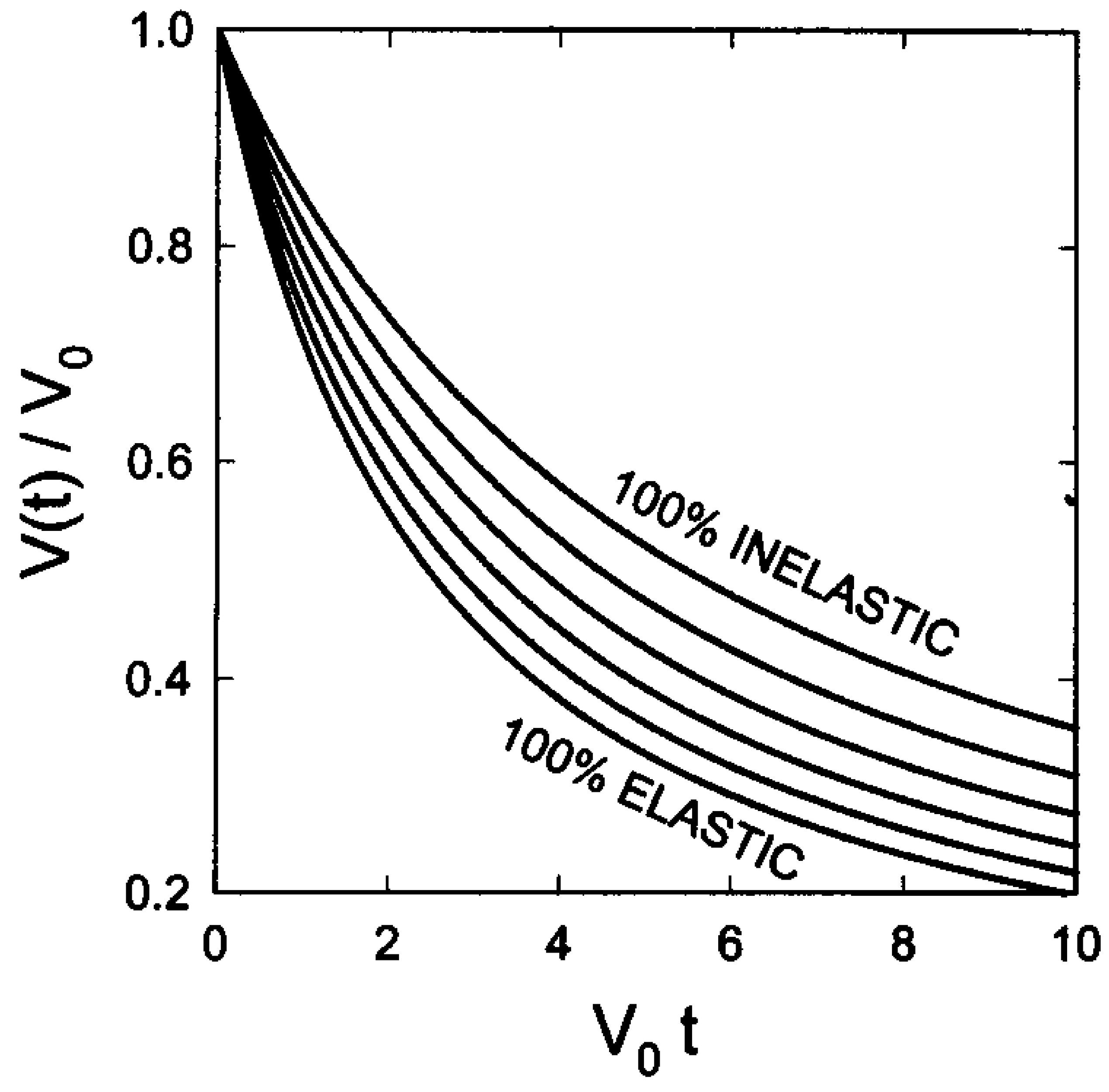

FIGURE 2. Velocity of body inside resistive medium as a function of time, for a mass ratio $r = 0.2$ and several restitution parameter values. From top to bottom: $\epsilon = 1$ (completely elastic), $0.8, 0.6, 0.4, 0.2$ and 0 (completely inelastic).

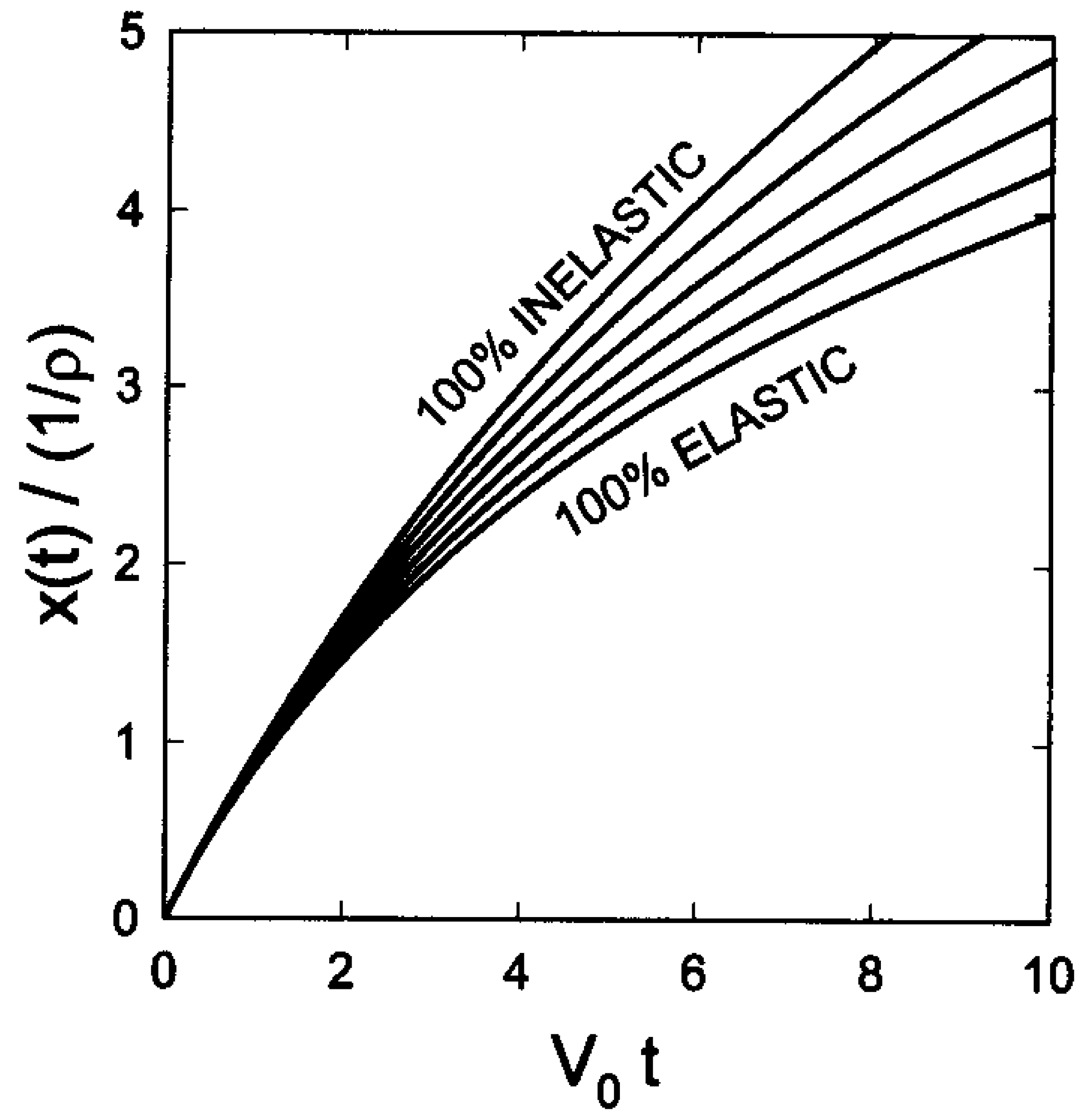

FIGURE 3. Distance travelled by body inside resistive medium, as a function of time, for the same parameter values as in Fig. 2.

ϵ^* as the restitution value characterized by $\gamma(r \sim 0, \epsilon^*) = \gamma(r = 1, \epsilon^*)$. A numerical computation using Eq. (9) gives $\epsilon^* = 0.5936624$. Second, let us compute the maximum fractional change in $\gamma(\epsilon^*, r)/\rho m$ as r sweeps from $r = 0$ to $r = 1$. To do that, we first find where the minimum value of $\gamma(\epsilon^*, r)/\rho m$ occurs, given by the solution r^* of $(d/dr)[\, \gamma(\epsilon^*, r)/\rho m \,] = 0$. We obtain $r^* = 0.515148$. The fractional change in γ will then be

$$\delta \equiv \left| 1 - \frac{\gamma(r^*, \epsilon^*)}{\gamma(r = 1, \epsilon^*)} \right| = 0.04925, \qquad (15)$$

i.e., the maximum fractional change along the $\epsilon = \epsilon^*$ curve is at most of 4.9%. Thus, for $\epsilon = \epsilon^* = 0.5936$, $\gamma/\rho m$ is nearly constant, which implies that the resistive coefficient γ is the same for either a large body mass or a small one (provided $r \leq 1$), and depends linearly on the medium mass density ρm. At higher inelasticity values, and all the way up to $r = 1$, the resistive coefficient decrease with r, as Fig. 1 shows.

Figure 2 shows the velocity of the body as a function of time for a particular mass ratio ($r = 0.2$) and for several restitution coefficients. The figure shows that the completely inelastic case is less effective in reducing the body's speed than the completely elastic case. The body's velocity decreases monotonically with time, going as $1/t$ at large times, as can be seen from Eq. (10). In Fig. 3 we show the distance travelled by the body inside the resistive medium, as a function of time, for the same parameter values as in Fig. 2. It clearly shows that the more inelastic the collision, the more distance the body travels, for a given time. This is because for the completely inelastic case, the transfer of momentum from the body to the medium particle is the smallest possible, while it is the largest for the completely elastic case. From Eq. (12) it can be proved that, for all values of the restitution parameter, $x(t)$ is always a monotonically increasing function of time which diverges logarithmically at large t values: the body does not have a finite stopping distance, no matter how small its initial kinetic energy. This is to be contrasted with the well-known case of a body subjected to a constant negative acceleration (as when we apply the brakes while driving a car), where there is a well-defined finite stopping distance. In general, for a resistive 'force law' of the type $F = -\gamma V^\alpha$, a finite stopping distance is always possible only if $\alpha \leq 1$ while a finite stopping time is possible for $\alpha < 1$ (see appendix).

The strong quadratic dependence of the resistive force on the velocity of the body for all values of elasticity, evidenced by our model, seems to suggest that it is due to the one-dimensionality of the system rather than to the actual nature of the interactions. The one-dimensionality constrains all collisions to be of the 'head-on' type, while in higher dimensional systems (and for reasonable shaped objects), most of the collisions would be of the "glancing" type thus reducing considerably the exchange of momentum and energy and leading to a weaker dependence of the body's velocity.

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Appendix

Stopping time, stopping distance and stopping power for $F = -\gamma V^\alpha$

Let us consider a hypothetical one-dimensional case where the resistive force on a body is given by $F = -\gamma V^\alpha$, with $\alpha > 0$. From Newton's equation we have,

$$M\ (d/dt)V = -\gamma V^\alpha, \qquad (16)$$

that is,

$$\int \frac{dV}{V^\alpha} = -\frac{\gamma}{M}\int dt. \tag{17}$$

We distinguish two cases:

(a) $\alpha = 1$. In that case, (17) leads to

$$V(t) = V_0 \exp\left[-\left(\frac{\gamma}{M}\right)t\right] \tag{18}$$

and

$$x(t) = \frac{MV_0}{\gamma}\left\{1 - \exp\left[-\left(\frac{\gamma}{M}\right)t\right]\right\}, \tag{19}$$

where we assume $x(0) = 0$. Thus, the stopping distance is MV_0/γ (even though the stopping time is infinite).

(b) $\alpha \neq 1$. In this case, integration of (17) leads to

$$V(t) = V_0\left[1 - \frac{\gamma}{M}\frac{(1-\alpha)t}{V_0^{1-\alpha}}\right]^{1/(1-\alpha)}, \tag{20}$$

which becomes zero at

$$t^* = \frac{V_0^{(1-\alpha)}}{\left(\frac{\gamma}{M}\right)(1-\alpha)}, \tag{21}$$

that is, a finite stopping time, provided $\alpha < 1$. From (20), and assuming $x(0) = 0$, we obtain

$$x(t) = \frac{V_0^{2-\alpha}}{\left(\frac{\gamma}{M}\right)(2-\alpha)} \times \left\{1 - \frac{1}{\left[1-(1-\alpha)V_0^{\alpha-1}\left(\frac{\gamma}{M}\right)t\right]^{\alpha/(1-\alpha)}}\right\}. \tag{22}$$

After inserting $t = t^*$ into (22), we obtain the stopping distance: $MV_0^{2-\alpha}/(2-\alpha)\gamma$.

On the other hand, the 'stopping power' for a system described by (16) will be

$$S(E) = -\frac{1}{\rho}\frac{dE}{dx} = -\frac{1}{\rho}MV\frac{dV}{dx} \tag{23}$$

but, (16) implies $dV/dx = -(\frac{\gamma}{M})V^{\alpha-1}$. Then,

$$S(E) = \frac{\gamma}{\rho}\left(\frac{2E}{M}\right)^{\alpha/2}. \tag{24}$$

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