Scalar solitons in a 4-dimensional curved space-time

J.A. González* and D. Sudarsky†
Instituto de Ciencias Nucleares, Universidad Nacional Autonoma de México
Apartado postal 70-543, México, D.F., Mexico
e-mail: *cervera@nuclecu.unam.mx, †sudarsky@nuclecu.unam.mx

Recibido el 10 de noviembre de 2000; aceptado el 13 de febrero de 2001

There is a theorem known as a Virial theorem that restricts the possible existence of non-trivial static solitary waves with scalar fields in a flat space-time with 3 or more spatial dimensions. This raises the following question: Does the analogous curved space-time version hold? We investigate the possibility of solitons in a 4-D curved space-time with a simple model using numerical analysis. We found that there exists a static solution of the proposed non linear wave equation. This proves that in curved space-time the possibilities of solitonic solutions is enhanced relative to the flat space-time case.

Keywords: Solitons

Existe un teorema conocido como un Teorema del Virial que restringe la posible existencia de ondas solitarias no triviales, con campos escalares en un espacio-tiempo plano con 3 o más dimensiones espaciales. Esto nos lleva a preguntarnos: ¿La versión análoga en espacio-tiempos curvos es válida? En este artículo investigamos la posibilidad de encontrar solitones en un espacio-tiempo 4-dimensional curvo, usando un modelo simple y con ayuda de un análisis numérico, encontrando que existe una solución estática de la ecuación de onda no lineal. Esto prueba que en un espacio-tiempo curvo es posible encontrar soluciones solitónicas a diferencia de un espacio-tiempo plano.

Descriptores: Solitones

PACS: 02.60.Cb; 02.60.Lj; 11.10.-z

1. Introduction

Solitons are special solutions of non-linear wave equations. The most relevant characteristic of solitons is that they are localized static solutions. The simplest example consist of a single scalar field ϕ in one spatial and one temporal dimensions. Perhaps the most famous one is the sine-Gordon soliton [1, 2].

At first sight, it can be thought that a wave equation with a single scalar field in more than three spatial dimensions with solitonic solutions can be found. However there is a Virial theorem which restricts this possibility. Here we are going to transcript that theorem and it's proof for the convenience of the reader [3]:

Theorem. *There are no non-trivial static solitary waves of systems with scalar fields when the space dimensionality is three or more and when the lagrangian has the form:*

$$\mathcal{L}(\mathbf{x},t) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - U\big(\phi(\mathbf{x},t)\big), \tag{1}$$

with $\phi = [\phi_i(\mathbf{x},t);\ i = 1,\dots,N]$ *a set of N coupled scalar fields in* D space plus one time dimensions, and $U(\phi(\mathbf{x},t))$ a positive definite *potential.*

Proof. *A static solution* $\phi(\mathbf{x})$ *obeys*

$$\nabla^2\phi = \frac{\partial U}{\partial \phi}(\mathbf{x}), \tag{2}$$

where ∇^2 *is the Laplacian in* D *dimensions. This equation clearly correspond to the extremum condition* $\delta W = 0$ *for the static energy functional*

$$W[\phi] \equiv \int d^{\mathrm{D}}x \left[\frac{1}{2}\nabla_i\phi\cdot\nabla_i\phi + U\big(\phi(\mathbf{x})\big)\right]$$
$$\equiv V_1[\phi] + V_2[\phi], \tag{3}$$

where the functionals V_1 *and* V_2 *stand for the two terms on the right-hand side. Note that not only W but also* V_1 *and* V_2 *are non-negative. Now, let* $\phi_1(\mathbf{x})$ *be a static solution. Consider the one-parameter family of configurations*

$$\phi_\lambda = \phi_1(\lambda\mathbf{x}). \tag{4}$$

It is easy to check that

$$W[\phi_\lambda] = V_1[\phi_\lambda] + V_2[\phi_\lambda]$$
$$= \lambda^{2-\mathrm{D}}V_1[\phi_1] + \lambda^{-\mathrm{D}}V_2[\phi_1]. \tag{5}$$

Since $\phi_1(\mathbf{x})$ *is an extremum of* $W[\phi]$, *it must in particular make* $W[\phi_\lambda]$ *stationary with respect to variations in* λ; *that is,*

$$\frac{d}{d\lambda}W[\phi_\lambda] = 0 \quad at\ \ \lambda = 1. \tag{6}$$

Differentiating Eq. (5) using Eq. (6) gives us

$$(2-\mathrm{D})V_1[\phi_1] = \mathrm{D}V_2[\phi_1]. \tag{7}$$

Since V_1 *and* V_2 *are non-negative Eq. (7) cannot be satisfied for* $\mathrm{D} \geq 3$ *unless* $V_1[\phi_1] = V_2[\phi_1] = 0$. *This means that* $\phi_1(\mathbf{x})$ *has to be space-independent and equal to one of the zeros of* $U[\phi]$. *This is just a trivial solution and the theorem precludes non-trivial space-dependent solutions. q.e.d.*

For the case of D = 2, Eq. (7) tell us that $V_2[\phi_1] = 0$. The simplest example of a solution of this kind is the non-linear $O(3)$ model [4–6] relevant in the description of the statical mechanics of an isotropic ferromagnet.

We are interested in the possible existence of a static solution of a system in a curved space-time in three spatial and one temporal dimensions. We will construct a simple model and look for solitonic solutions numerically.

2. The model

Consider the simplest case in a curved space-time. One scalar field whose equation of motion is

$$\partial^\mu \partial_\mu \phi - \frac{\partial V}{\partial \phi} = 0. \tag{8}$$

With a static potencial Eq. (8) is the Laplace equation,whose solutions take a maximum or minimum value only at the spatial boundaries. If we solve the Laplace equation in a space with a connected boundary or in a compact space, the solution in every point of the space will necessarily have the same value as in the boundary (*i.e.* a trivial solution).

So we are going to introduce a potential $V = (\phi^2 - 1)^2$ in analogy with the 1 + 1 dimensional case. Moreover, we consider the simplest kind of spatialy compact universe, the static Einstein universe:

$$ds^2 = -dt^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2). \tag{9}$$

Equation (8) in the space-time corresponding to Eq. (9) takes the form

$$2(\phi^2 - 1)2\phi = \frac{\partial^2\phi}{\partial\chi^2} - \frac{\partial^2\phi}{\partial t^2} + 2\cot\chi\frac{\partial\phi}{\partial\chi} + \sin^2\chi\cot\theta\frac{\partial\phi}{\partial\theta} + \sin^2\chi\frac{\partial^2\phi}{\partial\theta^2} + \sin^2\chi\sin^2\theta\frac{\partial^2\phi}{\partial\varphi^2}. \tag{10}$$

We are interested in static solutions and with spherical simetry so we have

$$\frac{d^2\phi}{d\chi^2} + 2\cot\chi\frac{d\phi}{d\chi} - 4\phi(\phi^2 - 1) = 0. \tag{11}$$

This is an ordinary differential equation of second order. Equation (11) can be separated in 2 ordinary equations of first order. Let

$$x_1 = \phi \quad \text{and} \quad x_2 = \frac{d\phi}{d\chi}, \tag{12}$$

so, Eq. (11) can be written as the following system:

$$x_2 = \frac{dx_1}{d\chi}, \tag{13}$$

$$0 = \frac{dx_2}{d\chi} + 2x_2\cot\chi - 4x_1(x_1^2 - 1). \tag{14}$$

These equations can be integrated by the Runge-Kutta method [7].

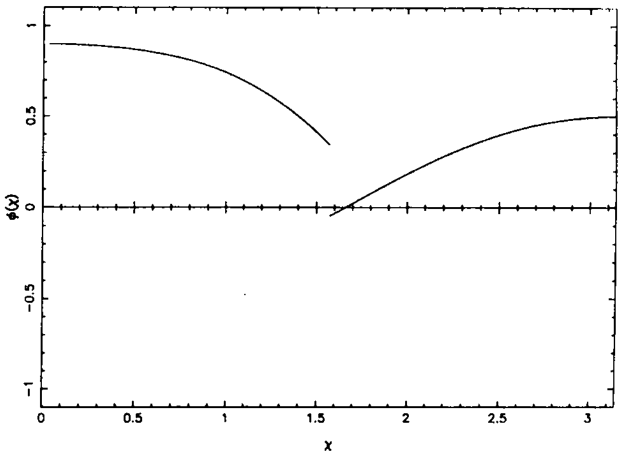

FIGURE 1. Integration for the initial values ϕ_a^0 and ϕ_b^0. Obviously $\phi_a(\pi/2) \neq \phi_b(\pi/2)$ and $\dot\phi_a(\pi/2) \neq \dot\phi_b(\pi/2)$.

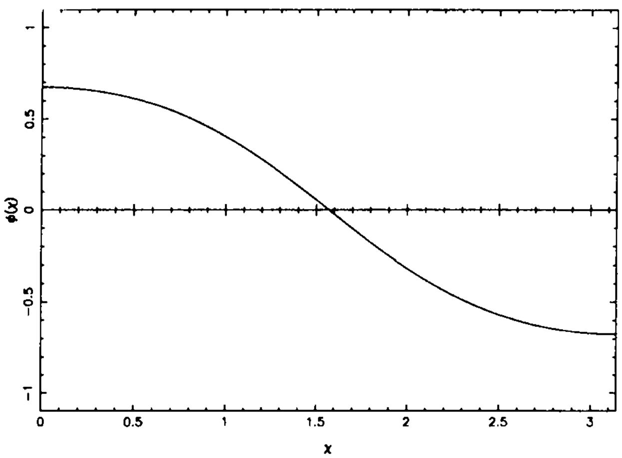

FIGURE 2. Integration after the application of the Newton-Raphson method. We can see that $\phi_a(\pi/2) = \phi_b(\pi/2)$ and $\dot\phi_a(\pi/2) = \dot\phi_b(\pi/2)$. This is a solution of the equation of motion (11).

We note that Eq. (11) is singular at $\chi = 0$ and $\chi = \pi$, so we use the following initial conditions:

$$\phi(0) = \phi_a^0, \quad \frac{d\phi(0)}{d\chi} = \dot\phi_a^0 = 0, \tag{15}$$

$$\phi(\pi) = \phi_b^0, \quad \frac{d\phi(\pi)}{d\chi} = \dot\phi_b^0 = 0. \tag{16}$$

So we have Eq. (11) with boundary conditions at the two end points. It can be solved using a "shooting to a middle point" method, with ϕ_a^0 and ϕ_b^0 as shooting parameters. The main idea of this method is the following: we perform an integration from $\chi = 0$ to $\chi = \pi/2$ obtaining $\phi_a(\pi/2)$. Also we integrate from $\chi = \pi$ to $\chi = \pi/2$ obtaining $\phi_b(\pi/2)$. Now, let us construct the following function ($f: R^2 \to R^2$):

$$F(\phi_a^0, \phi_b^0) = \left[\phi_a\left(\frac{\pi}{2}\right) - \phi_b\left(\frac{\pi}{2}\right), \dot\phi_a\left(\frac{\pi}{2}\right) - \dot\phi_b\left(\frac{\pi}{2}\right)\right]. \tag{17}$$

The first integration allows us to evaluate F obtaining in general $F(\phi_a^0, \phi_b^0) \neq 0$ (Fig. 1). We are interested in the zeros of this function, because at those points the solutions of the integration match ϕ and $\dot{\phi}$ in a smooth way. We used the Newton-Raphson method to find the zeros of a function [7], and the result is showed in Fig. 2.

3. Conclusion

We have shown that the theorem that precludes the existence of solitonic solutions to systems based in saclar field in $3+1$ or more flat space-time dimensions would be false if extended to the curve space-time case. We have done so by explicitly constructing one such solitonic solution of a simple model based on a single scalar field in the static Einstein universe.

This is another indication[(a)] that the interplay of curved space-time physics and soliton physics allow us for richer phenomena than each either of the two fields on its own.

Acknowledgments

This work was partially supported by DGAPA-UNAM grant No. IN121298 and CONACyT grant No. 32272E.

(a) The most widely known example of this phenomenon arose in the consideration of Einstein-Yang-Mills theory where solitonic solutions have been found while it know that these are not such solutions in the Yang-Mills theory in Minkowski space-time [8].

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