# Body motion in a resistive medium at temperature $T$

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We consider a macroscopic body propagating in a one-dimensional resistive medium, consisting of an ideal gas at temperature $T$. For a whole family of collisions with varying degree of inelasticity, we find an exact expression for the effective force on the moving body as a function of the body's speed and the value of the restitution coefficient. At low and high speeds it reduces to the well-known Stoke's and Newton's law, respectively.

*Keywords:* Air drag; collisions.

Se considera un cuerpo macroscópico propagándose en un medio resistivo unidimensional, consistente de un gas ideal a temperatura $T$. Para toda una familia de colisiones con diferente grado de inelasticidad, hallamos una expresión exacta para la fuerza efectiva sobre el cuerpo como función de la velocidad del cuerpo y del coeficiente de restitución. A bajas y altas velocidades, se reduce a la conocida ley de Stoke y Newton, respectivamente.

*Descriptores:* Roce viscoso; colisiones.

## 1. Introduction

When an object moves through a viscous medium, such as water or air, it experiences a resistive drag force. For small objects such as dust particles moving at low speeds, this drag force is proportional to the speed of the object. This is known as Stoke's law [1]. For larger objects such as airplanes, skidivers and baseballs moving at high speeds the drag force is approximately proportional to the square of the speed [2]. This limit is known as Newton's law. The general problem of determining the exact dependence of the drag force on the speed of an arbitrarily-shaped object moving at any speed, defies any closed form solution given its complex many-body character. A complete solution would have to take into account the detailed scattering process between the body and the particles composing the medium, the thermal properties of the medium, the presence of possible internal degrees of freedom of the body and local turbulence effects, etc. However, beneath all these complexities is basically the transfer of momentum and energy between the body and the medium particles. Therefore, is instructive to consider simplified "toy" models where one can track in detail the momentum and energy exchange between the body and its surrounding medium. This is realized at the expense of simplifying other factors such as the dimensionality of the system or the specific form of the interactions between the body and the medium. In this spirit, we present here an extension of a previous [3], simplified zero-temperature one-dimensional model, where we now include finite temperature effects. This immediately brings into the game a natural velocity scale not present in our previous model: the thermal speed. We obtain the resistive drag force as a function of the body's speed in **closed form** and find that, when the speed of the body is smaller than the thermal speed, the resistive force is *linearly* proportional to the speed of the body. On the contrary, when the speed of the body is greater then the thermal speed, the proportionality becomes *quadratic*.

## 2. The model

Let us consider a (macroscopic) body of mass $M$ propagating in a one-dimensional resistive medium modelled by an ideal gas in thermodynamic equilibrium at temperature $T$, characterized by a thermal speed $V_T \equiv \sqrt{kT/m}$, where $k$ is Boltzmann's constant and $m$ is the mass of a medium particle (Fig. 1). We assume the body to be truly macroscopic, like a baseball moving through air, or a falling rock. In other words, $M \gg m$ which allows us to make the following simplification: During a medium particle-body collision event, we will take the mass of the body to be essentially infinite. In this approximation the body is pictured as a massive, partially absorbing, moving "wall" colliding constantly with the medium particles. A reasonable assumption, if one considers that the mass ratio $m/M$ is of the order of $10^{-24}$ for a baseball moving through air. After each collision, the speed of the body is essentially unchanged, so the magnitude of the momentum transferred to the body is

$$\Delta p \approx (1 + \epsilon)m|V - v|,$$

where $V$ is the speed of the body, $v$ the speed of the medium particle and $\epsilon$ is the restitution coefficient for the body-particle collision. Thus, when $\epsilon = 1$ we have a completely elastic collision, where the magnitude of the relative body-particle velocity is conserved, while at $\epsilon = 0$, we have the case of a completely inelastic collision, where the particle is "absorbed" by the body after colliding. We also work in a

quasi-continuum approximation where an element of length $dx$ while "small", will contain a large number of medium particles.
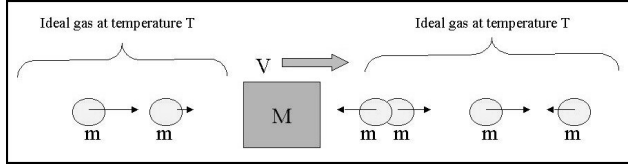


FIGURE 1. A macroscopic body of mass $M$ propagating inside a one-dimensional resistive medium composed of an ideal gas of particles of mass $m$, with $m \ll M$, in thermal equilibrium at temperature $T$. The body undergoes partially elastic collisions with the medium particles with restitution coefficient $\epsilon$.

Initially the body is given a speed $V_0$ (say, to the right), and we observe the system at a later time $t$, when the speed of the body is $V$. During the next time interval $dt$, the body will collide with particles coming from its left and right side. On the left side, only those particles that have speeds $v > V$ and are located closer than $(v - V)dt$ will collide with the body. The number of such particles is $dn_L = \rho dn(v)\Theta(v-V)(v-V)dt$. In a similar manner, the number of particles to the right of the body that will collide with the body during the interval $dt$ is $dn_R = \rho dn(v)\Theta(V-v)(V-v)dt$. Here $\rho$ is the particle number density, $\Theta(x)$ is the step function ($\Theta(x) = 1$, for $x > 0$, zero otherwise) and $dn(v)$ is the number of particles that have speeds in the interval $[v, v + dv]$: $dn(v) = g(v)dv$ where $g(v)$ is the thermal speed distribution, given by

$$g(v) = (1/\sqrt{2\pi})\frac{1}{V_T}\exp[-\frac{1}{2}(v/V_T)^2]. \quad (1)$$

The transfer of momentum per unit time coming from the medium to the *left* of the body, due to particles with speed in the interval $[v, v + dv]$ would then be

$$\frac{dP}{dt} = (1+\epsilon)m(v-V)g(v)\Theta(v-V)(v-V)dv. \quad (2)$$

By integrating over all speeds, we obtain the average effective force on the body from the left side:

$$F_{\text{left}} = \int_V^\infty \rho m(1+\epsilon)(v-V)^2 g(v)dv. \quad (3)$$

In a similar manner, the transfer of momentum per unit time coming from the medium to the *right* of the body, due to particles with speed in the interval $[v, v + dv]$ is

$$\frac{dP}{dt} = -(1+\epsilon)m(V-v)g(v)\Theta(V-v)(V-v)dv, \quad (4)$$

which implies that the average force on the body from the right side is

$$F_{\text{right}} = \int_{-\infty}^V \rho(1+\epsilon)(V-v)^2 g(v)dv. \quad (5)$$

The net average force $F$ on the body, along the direction of its initial velocity, is given by the difference between Eqs. (3) and (5):

$$F = -m\rho(1+\epsilon)\left[\int_{-\infty}^V \rho(1+\epsilon)(V-v)^2 g(v)dv\right.$$
$$\left. - \int_V^\infty \rho m(1+\epsilon)(v-V)^2 g(v)dv\right].$$

By inserting expression (1) for $g(v)$ and carrying out the integrations, we obtain

$$F = -m\rho(1+\epsilon)V_T^2\left\{\sqrt{\frac{2}{\pi}}\left(\frac{V}{V_T}\right)\exp\left[-\frac{1}{2}\left(\frac{V}{V_T}\right)^2\right]\right.$$
$$\left. + \left(1 + \left(\frac{V}{V_T}\right)^2\right)\text{Erf}\left(\frac{V}{\sqrt{2}V_T}\right)\right\} \quad (6)$$

This rather complex-looking expression is a bit deceiving since it depends on negative exponentials of $(V/V_T)^2$ which makes it very sensitive to whether $V/V_T < 1$ or $V/V_T > 1$. In other words, we expect two, well-defined behavior regimes, with a small crossover region near $V/V_T \approx 1$.

## 3. Results and discussion

As can be clearly seen from (6), the degree of inelasticity plays only a minor role, renormalizing the number density of the medium. Figure 2 is a log-log plot of the effective average force on the body as a function of the speed of the body, Eq.(6). As anticipated above, we note that except for a small vicinity around $V = V_T$, it consists of basically two straight lines with slopes of one and two, respectively. That is, at speeds smaller than the thermal speed $V_T$, the resistive force is proportional to the body's speed (Stoke's law); while for body's speeds greater than $V_T$, the resistive force becomes quadratic on the body's speed (Newton's law). These limits are easy to derive from Eq.(6): For $V \ll V_T$, $\text{Erf}(V/\sqrt{2}V_T) \approx \sqrt{2/\pi}(V/\sqrt{2}V_T)$ and $\exp[-(1/2)(V/V_T)^2] \approx 1$, which implies:

$$F \approx -\sqrt{8/\pi}m\rho(1+\epsilon)V_T V = -\rho(1+\epsilon)\sqrt{8mkT/\pi}\ V$$
$$V \ll V_T. \quad (7)$$

On the other hand, when $V \gg V_T$, $\text{Erf}(V/\sqrt{2}V_T) \approx 1$ and $\exp[-(1/2)(V/V_T)^2] \approx 0$. Thus, in this case one has:

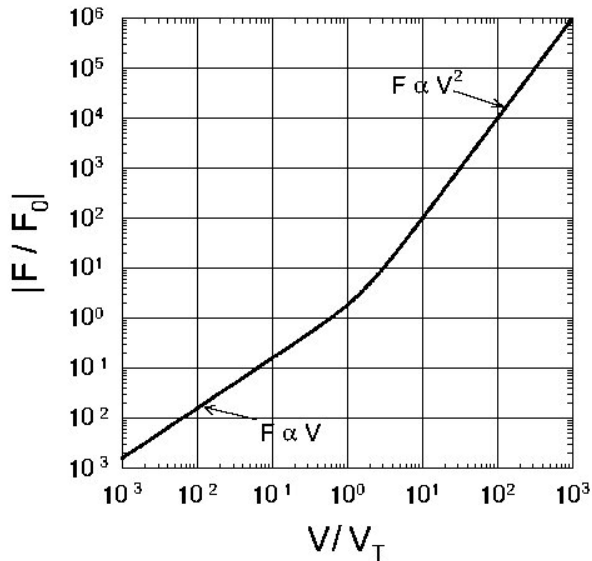$$F \approx -m\rho(1+\epsilon)V^2 \qquad V \gg V_T. \quad (8)$$

FIGURE 2. Effective average force on the macroscopic body, as a function of the body's speed. For speeds smaller (higher) than the thermal speed, the dependence is essentially linear (quadratic). The crossover region is confined to a small vicinity around $V_T$. ($F_0 \equiv m\rho(1+\epsilon)V_T^2$).

Let us now consider the issue of the stopping distance. For a medium at a finite temperature, the speed of the body decreases (on a macroscopic scale) as it moves through the medium and will eventually become smaller than the thermal speed. At that point, the resistive force becomes proportional to the speed, $F = -\beta V$. A simple integration then leads to an exponential decrease on $V$ and therefore, a *finite* stopping distance. If the medium is at zero temperature however, the resistive force is always quadratic with speed $F = \gamma V^2$ and, in that case, it can be easily proved that the stopping distance diverges logarithmically with time [3].

In summary, we have examined a simplified model of a macroscopic object propagating in a resistive one-dimensional medium modelled as an ideal gas at temperature $T$. For general inelastic collisions between the body and the medium particles, characterized by a restitution coefficient $\epsilon$, $0 \le \epsilon \le 1$, we have arrived at a closed-form solution for the resistive force in terms of the speed of the body. Below the thermal speed, this force is essentially linear in the body's speed, while above thermal speed, the dependence becomes quadratic.

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