

# Spheroidal functions revisited: matrix evaluation and generating functions

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A matrix method to evaluate the angular spheroidal functions is presented as an alternative to the orthodox method. Plane waves are identified as generating functions of the spheroidal functions showing their explicit expansions.

*Keywords:* Spheroidal functions; matrix evaluation; generating functions

Se presenta un método matricial para evaluar las funciones esféricas angulares como alternativa al método ortodoxo. Se identifican las ondas planas como funciones generadoras de las funciones esféricas, mostrando sus desarrollos explícitos.

*Descriptores:* Funciones esféricas; evaluación matricial; funciones generadoras

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## 1. Introduction

This work has been done in parallel to that on Mathieu functions [1], which is natural taking into account the connection between the Mathieu and spheroidal functions [2]. The methodology is applicable to both type of functions, but the results are specific to each one of them.

Chapter 21 on spheroidal wave functions by Lowan [2] is taken as the basic reference for the revisiting in this paper. Additional references are books by some of the authors who made original contributions to the mathematical understanding, the physical applications and the numerical computation of the functions themselves [3-7].

Mathieu functions and spheroidal functions have the common property of being solutions of Helmholtz equation, with the difference that the first ones are associated with elliptical coordinates in two dimensions while the second ones are associated with prolate/oblate spheroidal coordinates in three dimensions. Correspondingly, the angular Mathieu functions are represented as cosine or sine Fourier series in the hyperbolic coordinate, and the angular spheroidal functions are Legendre polynomial series in the hyperboloidal coordinates. The respective expansion coefficients are shown to satisfy three-term recurrence relations, which can be rewritten as equations involving infinite continued fractions in the characteristic numbers or eigenvalues of the separation constant. The orthodox method for the evaluation of the respective functions is implemented in two successive steps, first the numerical evaluation of each eigenvalue, and second the evaluation of the expansion coefficients.

The matrix method presented here as an alternative to evaluate the Mathieu functions and spheroidal functions has been used by our group in connection with the study of the hydrogen molecular ion in two and three dimensions, respec-

tively [8-10] in the context of matrix quantum mechanics. The advantage of this method in comparison with the orthodox one is that in the diagonalization process the eigenvalues and eigenvectors are simultaneously obtained, and their convergence and accuracy can be tested by increasing the size of the matrix.

The remaining of the manuscript is organized as follows. Section 2 shows the separation of the Helmholtz equation in both prolate and oblate spheroidal coordinates, obtaining the respective angular and radial equations for the spheroidal functions and their corresponding connections. Section 3 contains the matrix formulation of the eigenvalue problem for the separation constant in the angular equation. In Sec. 4, plane waves are identified as generating functions of the spheroidal functions, showing the explicit expansions in spheroidal waves. In the closing section some mathematical and physical applications are discussed.

## 2. Helmholtz equation in spheroidal coordinates

Prolate ( $1 \leq \xi < \infty$ ,  $-1 \leq \eta \leq 1$ ,  $0 \leq \varphi < 2\pi$ ) and oblate ( $0 \leq \zeta < \infty$ ,  $-1 \leq \chi \leq 1$ ,  $0 \leq \varphi < 2\pi$ ) spheroidal coordinates are defined by the respective transformation equations to cartesian coordinates

$$\begin{aligned} x &= c\sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos \varphi, \\ y &= c\sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin \varphi, \quad z = c\xi\eta, \end{aligned} \quad (1)$$

$$\begin{aligned} x &= c\sqrt{(\zeta^2 + 1)(1 - \chi^2)} \cos \varphi, \\ y &= c\sqrt{(\zeta^2 + 1)(1 - \chi^2)} \sin \varphi, \quad z = c\zeta\chi. \end{aligned} \quad (2)$$

Here  $c$  is the common semifocal distance and  $\varphi$  is the usual azimuthal angle. Confocal prolate spheroids are defined for each fixed value of  $\xi$  with an eccentricity  $1/\xi$ ; confocal two-sheet hyperboloids are defined for each fixed value of  $\eta$  with an eccentricity  $1/\eta$ . Confocal oblate spheroids are defined for each value of  $\zeta$  with an eccentricity  $1/\sqrt{\zeta^2 + 1}$ ; confocal one sheet hyperboloids are defined for each value of  $\chi$  with an eccentricity  $1/\sqrt{1 - \chi^2}$ , distinguishing between their upper and lower halves according to the intervals  $1 \geq \chi > 0$  and  $0 > \chi \geq -1$ , respectively.

Helmholtz equation in these coordinates has the respective forms

$$\left\{ \frac{1}{c^2(\xi^2 - \eta^2)} \left[ \frac{\partial}{\partial \xi}(\xi^2 - 1) \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}(1 - \eta^2) \frac{\partial}{\partial \eta} + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2}{\partial \varphi^2} \right] + k^2 \right\} \psi(\xi, \eta, \varphi) = 0, \quad (3)$$

$$\left\{ \frac{1}{c^2(\zeta^2 + \chi^2)} \left[ \frac{\partial}{\partial \zeta}(\zeta^2 + 1) \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \chi}(1 - \chi^2) \frac{\partial}{\partial \chi} + \frac{\zeta^2 + \chi^2}{(\zeta^2 + 1)(1 - \chi^2)} \frac{\partial^2}{\partial \varphi^2} \right] + k^2 \right\} \psi(\zeta, \chi, \varphi) = 0. \quad (4)$$

Both equations are separable admitting factorable solutions:

$$\psi(\xi, \eta, \varphi) = \Xi(\xi)H(\eta)\Phi(\varphi), \quad (5)$$

$$\psi(\zeta, \chi, \varphi) = Z(\zeta)X(\chi)\Phi(\varphi). \quad (6)$$

The azimuthal equation is common to both cases:

$$\frac{d^2 \Phi}{d\varphi^2} = -m^2 \Phi, m = 0, \pm 1, \pm 2, \dots \quad (7)$$

The hyperboloidal equations for the respective cases take the forms

$$\left[ \frac{d}{d\eta}(1 - \eta^2) \frac{d}{d\eta} - \frac{m^2}{1 - \eta^2} - k^2 c^2 \eta^2 \right] H(\eta) = -\lambda H(\eta), \quad (8)$$

$$\left[ \frac{d}{d\chi}(1 - \chi^2) \frac{d}{d\chi} - \frac{m^2}{1 - \chi^2} + k^2 c^2 \chi^2 \right] X(\chi) = -\lambda X(\chi), \quad (9)$$

differing only in the sign of the quadratic term in the hyperboloidal coordinate. Equation (9) may be obtained from Eq. (8) with the transformations  $\eta \rightarrow \chi$  and  $kc \rightarrow \mp ikc$ .

The corresponding spheroidal equations are

$$\left[ \frac{d}{d\xi}(\xi^2 - 1) \frac{d}{d\xi} - \frac{m^2}{\xi^2 - 1} + k^2 c^2 \xi^2 \right] \Xi(\xi) = \lambda \Xi(\xi), \quad (10)$$

$$\left[ \frac{d}{d\zeta}(\zeta^2 + 1) \frac{d}{d\zeta} + \frac{m^2}{\zeta^2 + 1} + k^2 c^2 \zeta^2 \right] Z(\zeta) = \lambda Z(\zeta), \quad (11)$$

differing in the signs of the binomials and of the second term. Equation (11) may be obtained from Eq. (10) with the transformations  $\xi \rightarrow \pm i\zeta$  and  $kc \rightarrow \mp ikc$ .

In the above equations,  $m$  and  $\lambda$  are the separation constants and play the role of eigenvalues for the respective operators. Equations (8) and (10) are identical, but their solutions cover different domains  $-1 \leq \eta \leq 1$  and  $1 \leq \xi < \infty$  for the hyperboloidal and spheroidal coordinates, respectively.

Similarly, Eq. (9) becomes Eq. (11) under the transformation  $\chi \rightarrow \pm i\zeta$ , and their solutions are defined in the respective domains  $-1 \leq \chi \leq 1$  and  $0 \leq \zeta < \infty$ .

We start by focussing our attention on Eqs. (8) and (9) for the angular spheroidal functions. The first two terms correspond to the Legendre operator, and all the terms are even in the hyperboloidal coordinate. This suggests solutions expressed as expansions of Legendre polynomials or Legendre functions of the second kind, which for the prolate spheroidal functions take the forms for the respective kinds:

$$S_{mn}^{(1)}(kc, \eta) = \sum_{r=0,1}^{\infty} {}_r d_r^{mn}(kc) P_{m+r}^m(\eta), \quad (12)$$

$$S_{mn}^{(2)}(kc, \eta) = \sum_{r=-\infty}^{\infty} {}_r d_r^{mn}(kc) Q_{m+r}^m(\eta). \quad (13)$$

The summations extend over even or odd values of  $r$ .

The substitution of Eq. (12) in Eq. (8) leads to three-term recurrence relations for the expansion coefficients Eq. 21.7.3 in Ref. 2. Like in the case of Mathieu functions, these relations can be transformed into an infinite continued fraction equation for the eigenvalues  $\lambda_{mn}$ , Eqs. 21.7.4 in Ref. 2. The orthodox method to evaluate the eigenvalues and expansion coefficients of the spheroidal functions of Eq. (12) is the adaptation of the method introduced by Ince for the Mathieu functions [11], involving the refinement of approximate numerical values of each eigenvalue, the evaluations of the ratios of consecutive expansion coefficients, and the evaluation of the coefficients themselves taking into account the adopted normalization convention, Eqs. 21.7.7–21.7.16 in Ref. 2.

The angular oblate spheroidal Eq. (9) has also solutions of the type of Eqs. (12) and (13) with the replacements  $\eta \rightarrow \chi$  and  $kc \rightarrow -ikc$  discussed after Eqs. (8) and (9). The corresponding expansion coefficients satisfy the same recurrence relations with the change of sign in  $k^2 c^2$ .

The radial prolate spheroidal Eq. (10) has also solutions of the type of Eqs. (12) and (13) with the substitution  $\eta \rightarrow \xi$  and corresponding change of domain. It is recognized that Eq. (10) has a regular singular point at  $\xi = 1$ , which can be removed through a factor  $(\xi^2 - 1)^{m/2}$  in the solution  $\Xi(\xi)$ . Then the remaining factor can be represented in terms of series of radial spherical Bessel functions. As an illustration, here we simply quote 21.9.1. from Ref. 2:

$$R_{mn}^{(p)}(kc, \xi) = \left\{ \sum_{r=0,1}^{\infty} \frac{{}_r (2m+r)!}{r!} d_r^{mn} \right\}^{-1} \left( \frac{\xi^2 - 1}{\xi^2} \right)^{m/2} \times \sum_{r=0,1}^{\infty} {}_r i^{r+m-n} \frac{(2m+r)!}{r!} d_r^{mn} Z_{m+r}^{(p)}(kc\xi), \quad (14)$$

where

$$Z_l^{(1)}(z) = j_l(z), \quad Z_l^{(2)}(z) = y_l(z) \quad (15)$$

are spherical Bessel functions of order  $l = r + m$ , of the first and second kinds, respectively. The expansion coefficients in Eq. (14)  $d_r^{mn}$  are the same as in Eqs. (12) and (13).

The radial oblate spheroidal functions, solutions of Eq. (11) can be obtained in two alternative ways. The first one is through the substitution  $\chi \rightarrow -i\zeta$  in the angular solution of Eq. (9). And the second one is through the substitutions  $\xi \rightarrow i\zeta$  and  $kc \rightarrow -ikc$  in the radial solutions of Eq. (10).

### 3. Matrix evaluation of angular spheroidal functions

Here we formulate the matrix solutions of both Eqs. (8) and (9) following the standard methods of quantum mechanics. We adapt the expansion of Eq. (12) to the normalized basis of associated Legendre functions:

$$|lm\rangle = \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}} P_l^m, \quad (16)$$

with  $l = r + m$ . The new expansion is

$$\left[ l(l+1) \pm k^2 c^2 \frac{(2l+3)(l+m)(l-m) + (2l-1)(l+m+1)(l-m+1)}{(2l+1)(2l-1)(2l+3)} - \lambda \right] \delta_{l,l} \\ \pm k^2 c^2 \left[ \frac{1}{2l-1} \sqrt{\frac{(l+m)(l+m-1)(l-m)(l-m-1)}{(2l+1)(2l-3)}} \delta_{l,l-2} \right. \\ \left. + \frac{1}{2l+3} \sqrt{\frac{(l+m+1)(l+m+2)(l-m+1)(l-m+2)}{(2l+1)(2l+5)}} \delta_{l,l+2} \right] = 0, \quad (20)$$

where the upper (lower) sign in the  $k^2 c^2$  terms are for the prolate (oblate) cases. Therefore, the matrix is tridiagonal and symmetric involving only even or only odd values of  $l$  or  $r$ . Its contents are equivalent to the three-term recurrence relation between the  $d_r^{mn}$  coefficients Eqs. 21.7.3 in Ref. 2. The advantage of using the normalized basis of Eq. (16) is the symmetry of the matrix of Eq. (20). The diagonalization of such a matrix produces both the eigenvalues  $\lambda_{mn}$  and the eigenvectors  $c_l^{mn}$  simultaneously. This can be contrasted with the successive steps for their individual determination in the orthodox method.

For  $c = 0$  the spheroids become spheres, the matrix is diagonal, the eigenvalues are  $\lambda = l(l+1)$  and the spheroidal functions become Legendre polynomials. For small values of  $kc$ , quantum perturbation theory can be applied to obtain

$$|mn\rangle = \sum_{l=|m|}^{\infty} c_l^{mn} |lm\rangle \quad (17)$$

differing from that of Eq. (12) in the use of orthonormalized bases. The connection between the expansion coefficients in Eqs. (12) and (17) is

$$d_r^{mn} = c_l^{mn} \frac{N_{lm}}{\bar{N}_{mn}}, \quad (18)$$

where  $N_{lm}$  is the normalization coefficient of the Legendre polynomials in Eq. (16) and  $\bar{N}_{mn}$  is the normalization coefficient of the angular spheroidal functions for which there are several conventions [1-7]. In particular, that of Meixner and Schäferke [5] is such that

$$\int_{-1}^1 \left| S_{mn}^{(1)}(kc, \eta) \right|^2 d\eta = \frac{2}{2n+1} \frac{(n+|m|)!}{(n-|m|)!}. \quad (19)$$

Then the normalization constant  $\bar{N}_{mn}$  has the same form as that of  $N_{lm}$  with the replacement of  $l$  by  $n$ .

The quadratic term in Eqs. (8) and (9) can be written as the combination  $(2P_2 + P_0)/3$ , with the result that its matrix elements in the basis of Eq. (16) have the selection rules  $l' = l - 2, l + 2$  and  $l$ , respectively. The other terms in both equations are diagonal. Then the corresponding matrices are

power series expansions for the eigenvalues Eqs. 21.7.5 in Ref. 2, and eigenvectors.

The size of the matrix is chosen according to the size of  $kc$ . The tabulations for the eigenvalues  $\lambda_{mn}(kc)$  in [2] cover the ranges of  $(kc)^2$  from 0 to 16 and for  $(kc)^{-1}$  from 0.25 to 0, and for  $m = 0, 1, 2$  and  $n = m, m+1, m+2, m+3, m+4$ . For the first range of  $(kc)^2$  matrices of  $10 \times 10$ , and for the second range of  $(kc)^{-1}$  from 0.25 to 0.01, matrices of  $60 \times 60$ , using the Jacobi diagonalization routine, produce eigenvalues matching and improving those tabulated in Ref. 2. The eigenvectors obtained in the corresponding diagonalizations also lead to reliable and accurate angular and radial prolate/oblate spheroidal functions.

### 4. Plane waves as generating functions of spheroidal functions

Plane waves

$$\psi(x, y, z) = e^{i\vec{k}\cdot\vec{r}} = e^{i(k_x x + k_y y + k_z z)} \tag{21}$$

and spherical waves

$$\psi(r, \theta, \varphi) = j_l(kr)Y_{lm}(\theta, \varphi) \tag{22}$$

are regular solutions of Helmholtz equation in three dimensions, satisfying orthogonality and completeness properties in their common domain  $(-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty)$  and  $(0 < r < \infty, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$ . These properties make possible the expansion of a plane wave in spherical waves, Eq. 10.1.47 in Ref. 2,

$$\begin{aligned} e^{i\vec{k}\cdot\vec{r}} &= \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\hat{k} \cdot \hat{r}) \\ &= \sum_{l=0}^{\infty} i^l j_l(kr) 4\pi \sum_{m=-l}^l Y_{lm}^*(\theta_k, \varphi_k) Y_{lm}(\theta, \varphi), \end{aligned} \tag{23}$$

where the addition theorem of the spherical harmonics is used in going between the last two lines. Equation (23) can also be interpreted as giving a generating function for the spherical Bessel functions.

As in the case of the Mathieu functions [1], there are not explicitly recognized generating functions for the spheroidal functions, [3-7]. Since the spheroidal waves

$$\psi(\xi, \eta, \varphi) = R_{mn}^{(1)}(kc, \xi) S_{mn}^{(1)}(kc, \eta), \tag{24}$$

$$\psi(\zeta, \chi, \varphi) = R_{mn}^{(1)}(-ikc, i\zeta) S_{mn}^{(1)}(-ikc, \chi), \tag{25}$$

are regular solutions of Helmholtz equation forming complete orthogonal sets in the common domain shared with the waves from Eqs. (21) and (22), we expect the existence of generating functions for the spheroidal functions as well.

Next, we proceed to justify such an expectation by finding the expansions of some plane waves in spheroidal waves. Eqs. (1) and (2) for the coordinate transformations, the associations of polar angles to the hyperboloidal coordinates  $\eta = \cos \theta_\eta$  and  $\chi = \cos \theta_\chi$ , and Eq. (23) are the basis to construct some of the expansions.

The wave propagating in the direction of the  $z$  axis can be written as

$$e^{ikz} = e^{ikc\xi \cos \theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kc\xi) P_l(\eta). \tag{26}$$

Since both the Legendre polynomials and the angular spheroidal functions form orthogonal complete sets of functions, the inverse transformation of Eq. (12) is

$$P_{m+r}^m(\eta) = \sum_{n=0,1}^{\infty} \bar{d}_n^{mr}(kc) S_{mn}^{(1)}(kc, \eta). \tag{27}$$

The respective expansion coefficients are related through the common overlap integral and the respective normalization integrals:

$$\begin{aligned} d_r^{mn}(kc) &= \frac{2(r+2m)!}{(2m+2r+1)r!} \\ &\times \int_{-1}^1 P_{m+r}^m(\eta) S_{mn}^{(1)}(kc, \eta) d\eta, \end{aligned} \tag{28}$$

$$\bar{d}_n^{mr}(kc) = \frac{1}{\bar{I}_{mn}} \int_{-1}^1 S_{mn}^{(1)}(kc, \eta) P_{m+r}^m(\eta) d\eta \tag{29}$$

$$\therefore \bar{d}_n^{mr}(kc) = \frac{I_{m+r}^m}{\bar{I}_{mn}} d_r^{mn}(kc) \tag{30}$$

In Eq. (26)  $m = 0$ , so that substituting this value in Eqs. (30), (27) and (26) successively, the result is

$$\begin{aligned} e^{ikc\xi\eta} &= \sum_{l=0}^{\infty} i^l (2l+1) j_l(kc\xi) \\ &\times \sum_{n=0,1}^{\infty} \frac{I_l^0}{\bar{I}_{0n}} d_l^{0n}(kc) S_{0n}^{(1)}(kc, \eta). \end{aligned} \tag{31}$$

The order of the summations can be exchanged, the explicit values of the normalization integrals  $I_l^0 = 2/(2l+1)$  and  $\bar{I}_{0n} = 2/(2n+1)$  can be substituted, and Eq. (14) is taken into account anticipating the identification of the radial spheroidal functions, leading to

$$\begin{aligned} e^{ikc\xi\eta} &= \sum_{n=0}^{\infty} i^n (2n+1) S_{0n}^{(1)}(kc\eta) \\ &\times \sum_{l=0,1}^{\infty} i^{l-n} d_l^{0n}(kc) j_l(kc\xi) \\ &= \sum_{n=0}^{\infty} i^n (2n+1) S_{0n}^{(1)}(kc, \eta) \\ &\times \left\{ \sum_{l=0,1}^{\infty} i^{l-n} d_l^{0n}(kc) \right\} R_{0n}^{(1)}(kc, \xi). \end{aligned} \tag{32}$$

The corresponding expansion in oblate spheroidal wave function is

$$\begin{aligned} e^{ikz} = e^{ikc\zeta\chi} &= \sum_{n=0}^{\infty} i^n (2n+1) S_{0n}^{(1)}(-ikc, \chi) \\ &\times \left\{ \sum_{l=0}^{\infty} i^l d_l^{0n}(-ikc) \right\} R_{0n}^{(1)}(-ikc, i\zeta). \end{aligned} \tag{33}$$

Next, we consider a plane wave propagating in the direction of the  $x$  axis ( $\theta_k = \pi/2, \varphi_k = 0$ ), which can be written in prolate spheroidal coordinates as

$$e^{ikx} = e^{ikc\sqrt{\xi^2-1} \sin \theta_\chi \cos \varphi} = \sum_{l=0}^{\infty} i^l j_l(kc\sqrt{\xi^2-1}) 4\pi \sum_{m=-l}^l Y_{lm}^*\left(\frac{\pi}{2}, 0\right) Y_{lm}(\theta_\eta, \varphi). \tag{34}$$

The spherical harmonic function in the angular spheroidal coordinates  $\theta_\chi$  and  $\varphi$  can be written in terms of the associated Legendre polynomials and in turn in terms of the angular spheroidal functions

$$\begin{aligned} Y_{m+r,m}(\theta_\chi, \varphi) &= \frac{e^{im\varphi}}{\sqrt{2\pi}} N_{m+r,m} P_{m+r}^m(\eta) \\ &= \frac{e^{im\varphi}}{\sqrt{2\pi}} N_{m+r,m} \sum_{n=m}^{\infty} {}_r d_n^{mr}(kc) S_{mn}^{(1)}(kc, \eta) \\ &= \frac{e^{im\varphi}}{\sqrt{2\pi}} N_{m+r,m} \sum_{n=m}^{\infty} {}_r \frac{I_l^m}{\bar{I}_{mn}} d_r^{mn}(kc) S_{mn}^{(1)}(kc, \eta), \end{aligned} \tag{35}$$

where Eqs. (27) and (30) have been used in the successive steps. The spherical harmonic in the angles of the propagation vector is also written

$$Y_{r+m,m}^*\left(\frac{\pi}{2}, 0\right) = \frac{N_{m+r,m}}{\sqrt{2\pi}} P_{m+r}^m(0). \tag{36}$$

The product of the normalization constants and the normalization integral associated with the Legendre polynomials in Eqs. (35) and (36) reduces to unity, when substituted in Eq. (34), which becomes

$$e^{ikc\sqrt{(\xi^2-1)(1-\eta^2)} \cos \varphi} = 2 \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} e^{im\varphi} S_{mn}^{(1)}(kc, \eta) \sum_r {}_r i^{m+r} \frac{d_r^{mn}(kc)}{\bar{I}_{mn}} P_{m+r}^m(0) j_{m+r}(kc\sqrt{\xi^2-1}). \tag{37}$$

When the normalization integral of the angular spheroidal functions, Eqs. (19) are used, Eq. (37) can be written as

$$\begin{aligned} e^{ikc\sqrt{(\xi^2-1)(1-\eta^2)} \cos \varphi} &= \sum_n \sum_m i^n (2n+1) \frac{(n-|m|)!}{(n+|m|)!} S_{mn}^{(1)}(kc, \eta) e^{im\varphi} \sum_r i^{m+r-n} P_{m+r}^m(0) d_r^{mn}(kc) j_{m+r}(kc\sqrt{\xi^2-1}) \end{aligned} \tag{38}$$

In this case the expansion of the plane wave has served to generate a representation of the radial spheroidal wave function  $R_{mn}^{(1)}(kc, \xi)$  via the last sum over the index  $r$  as a superposition of the spherical Bessel functions with the argument

$kc\sqrt{\xi^2-1}$ , as an alternative to that of Eq. (14). They are equivalent up to a normalization factor.

The corresponding expansion in oblate spheroidal function is

$$\begin{aligned} e^{ikx} = e^{ikc\sqrt{(\zeta^2+1)(1-\chi^2)} \cos \varphi} &= \sum_n \sum_m i^n (2n+1) \frac{(n-|m|)!}{(n+|m|)!} S_{mn}^{(1)}(-ikc, \chi) e^{im\varphi} \sum_r i^{m+r-n} P_{m+r}^n(0) d_r^{mn}(-ikc) j_{m+r}(kc\sqrt{\zeta^2+1}) \end{aligned} \tag{39}$$

### 5. Discussion

In Sec. 2 the separation of the Helmholtz equation in prolate and oblate spheroidal coordinates and some of the properties of the radial and angular spheroidal wave functions and their evaluation via the orthodox method have been briefly reviewed. Then in Sec. 3 the matrix evaluation of the spheroidal wave functions was formulated through the construction of the eigenvalue matrix Eq. (20). Its diagonalization provides the numerical values of the eigenvalues  $\lambda_{mn}$  and the eigenvectors  $c_l^{mn}$  of Eq. (16). The standard coeffi-

cients  $d_r^{mn}$  of Eq. (12) can be obtained by using Eq. (17). In Section 4 plane waves along the z-axis and along the x-axis have been identified as generating functions of the spheroidal wave functions, with  $m = 0$  Eqs. (32) and (33), and with  $m \neq 0$  Eqs. (38) and (39), respectively.

As an illustration of a mathematical application of the generating functions, Eq. (31) can be used as the starting point to obtain integral representations of the radial spheroidal functions. By separating its real and imaginary

parts and using the orthogonality of the angular spheroidal functions, the following representations are obtained:

$$R_{0,2s}^{(1)}(kc, \xi) = (-)^s 2 \left\{ \sum_{l=0}^{\infty} {}_l d_l^{0,2s}(kc) \right\}^{-1} \int_{-1}^1 d\eta S_{0,2s}^{(1)}(kc, \eta) \cos(kc\xi\eta), \quad (40)$$

$$R_{0,2s+1}^{(1)}(kc, \xi) = (-)^s 2 \left\{ \sum_{l=1}^{\infty} {}_l d_l^{0,2s+1}(kc) \right\}^{-1} \int_{-1}^1 d\eta S_{0,2s+1}^{(1)}(kc, \eta) \sin(kc\xi\eta). \quad (41)$$

Similar integral representations for  $R_{mn}^{(1)}(kc, \xi)$  can be obtained from Eq. (38):

$$\begin{aligned} & \sum_r i^{m+r-n} P_{m+r}(0) d_r^{mn}(kc) j_{m+r}(kc\sqrt{\xi^2-1}) \\ &= \frac{i^n}{2\pi} \int_{-1}^1 \int_0^{2\pi} d\eta d\varphi S_{mn}^{(1)}(kc, \eta) \\ & \quad \times e^{-im\varphi} e^{ikc\sqrt{(\xi^2-1)(1-\eta^2)} \cos\varphi}. \quad (42) \end{aligned}$$

Also the counterparts of Eqs. (40) – (42) for the corresponding oblate spheroidal functions can be constructed, either by analytical continuation or starting from Eqs. (33) and (39).

Spheroidal wave functions are useful to describe acoustic and electromagnetic waves radiated by sources with spheroidal shapes, and to describe the scattering of plane waves of either type by spheroidal obstacles [4]. Our group is presently investigating electron scanning tunnelling microscopy by modelling the tip and sample surfaces as confocal hyperboloidal electrodes and using spheroidal electronic wave functions [12].

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