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Complete description of weakly coupled chaotic subsystems

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We study the dynamics of one–dimensional lattices of weakly coupled maps of $\mathbb{R}$. The local dynamics has an invariant hyperbolic set. Moreover, the trajectories from non expanding (and weakly expanding) points go to infinity (for local dynamical system). Under these assumptions we show that, if the coupling is weak enough, the extended system has similar dynamics.

Keywords: Coupled map lattices, chaos, partial differential equations.

Estudiamos la dinámica de enrejados unidimensionales de mapeos de $\mathbb{R}$ acoplados débilmente. La dinámica local tiene un conjunto hiperbólico invariante. Además, las trayectorias de puntos no expansivos (y débilmente expansivos) van a infinito (para el sistema dinámico local). Bajo estos supuestos mostramos que, si el acoplamiento es suficientemente débil, el sistema extendido tiene una dinámica similar.

Descriptores: Enrejados de mapeos acoplados, caos, ecuaciones diferenciales.

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1. Introduction

Coupled map lattices (CMLs) serve as one of the most useful and powerful instruments for understanding the dynamics of spatially extended systems. The main activity in this field is directed toward the study of CMLs for small values of the spatial interactions; this is the case, for example, when CMLs with diffusive coupling are used to describe some lattice spin models of statistical mechanics in the region of high temperature [4].

For active homogeneous media, CMLs are described by equations of the form

$$u_s(t+1) = f\big(u_s(t)\big) + \gamma F\big(\{u_{s'}(t)\}_{|s'-s|\leqslant r}\big), \qquad (1)$$

where t is the discrete time coordinate, s is the discrete space coordinate, and $u_s(t)$ is a characteristic of the medium (for example, its density, distribution of temperature, etcetera). We suppose that the local map f and the coupling map F (of range r) are smooth maps; we also suppose that the coupling parameter γ is sufficiently small.

A natural source of CMLs are discrete versions of partial differential equations of evolution type. They arise while modeling PDE's by computer. As an example, let us mention the nonlinear reaction–diffusion equation

$$\frac{\partial u}{\partial t} = h(u) + \kappa A \Delta u, \qquad (2)$$

where $u = u(x,t)$ is a function of two variables (the space coordinate x and time t) with values in the d–dimensional Euclidean space $\mathbb{R}^d$; A is the coupling matrix,

$$\Delta u = \left(\frac{\partial^2 u_1}{\partial x^2}, \dots, \frac{\partial^2 u_d}{\partial x^2}\right),$$

and κ is the diffusion coefficient. Equation (2) describes a large variety of phenomena in different fields. Examples are heat conductivity, chemical diffusion processes, enzyme kinetics, and propagation of voltage impulses through nerve axons.

One can obtain a number of well known particular cases of reaction–diffusion Eq. (2) by an appropriate choice of the nonlinear term h. Among them are:

1. The Kolmogorov–Petrovsky–Piskunov equation for which the nonlinear term is the quadratic polynomial,

   $$h(u) = \alpha u(1-u),$$

   where $\alpha > 0$ is a parameter.

2. The Huxley equation for which the nonlinear term is the cubic polynomial,

   $$h(u) = \alpha u(1-u)(u-a),$$

   where $0 < a < 1$ and $\alpha > 0$ are parameters.

3. The FitzHugh–Nagumo equation for which the nonlinear term is the two–dimensional map of the plane,

   $$h(u,v) = (a\varphi(u) - bv, cu - dv).$$

   Here $\varphi(u) = u(u-\theta)(1-u)$ with $\theta \in (0,1)$ and $a, b, c, d > 0$ are real parameters.

For some of this PDEs a particular and even full qualitative picture of solution is known (see for example Ref. 6). But, for CMLs there were no completely described examples until now.

In this work we provide a full topological description of all possible orbits for some class of CML's. Explicitly, in

Refs. 1 and 3 it was shown that if the coupling is weak and the local map f satisfies some chaotic properties (see below), then there is an invariant subset on which all orbits can be described in terms of symbolic dynamics; we show that under some additional conditions, orbits starting from other points go to infinity. In order to do that, we consider the equation, for space–time configurations,

$$(\mathcal{G}_\gamma(u))_{st} \doteq f(u_{st}) + \gamma F(\{u_{st}\}^r) - u_{s,t+1} = 0$$

where $u = \{u_{s,t}\}$, $s \in \mathbb{Z}$ is space coordinate and $t \in \mathbb{N}$ is time (the explicit meaning and characteristics of the parameters and functions in the previous equation are given bellow). We investigate this equation without coupling ($\gamma = 0$) and then apply known results of functional analysis. Let us remark that, in principal, such approach also allow us to prove the existence of the invariant set.

The article is organized as follows. In Sec. 2 we give necessary definitions and formulate the main theorem (theorem 2.2). In Sec. 3 we formulate some results about the dynamics of a particular local map, which satisfies the hypothesis of the main theorem. In Sec. 4 we prove theorem 2.2. Finally, in Sec. 5 we present the concluding remarks for this work.

2. Weakly coupled map lattices

In the article we will consider coupled maps as dynamical systems. The phase space of them will be the set of bi–infinite uniform bounded sequences

$$\ell^\infty \doteq \{\, u = \{u_s\}_{s\in\mathbb{Z}} \;:\; u_s \in \mathbb{R};\ \|\, u\,\|_\infty < \infty \,\},$$

with

$$\|\, u\,\|_\infty = \sup_{s\in\mathbb{Z}} |u_s|.$$

The evolution operator $\mathcal{F}_{f,F} : \ell^\infty \to \ell^\infty$ will be defined by

$$(\mathcal{F}_{f,F}(u))_s = f(u_s) + \gamma F(\{u_s\}^r), \tag{3}$$

where $\{u_s\}^r = (u_{s-r}, \ldots, u_{s+r})$ and $r \in \mathbb{Z}^+$ is the length of interaction, $f : \mathbb{R} \to \mathbb{R}$ is the local map, $F : \mathbb{R}^{2r+1} \to \mathbb{R}$ is the coupling map and $\gamma \in \mathbb{R}$ is a parameter controlling the coupling.

In what follows we will assume that f is C^1–smooth and F is C^2–smooth. It is easy to see that $\mathcal{F}_{f,F}(\ell^\infty) \subset \ell^\infty$; moreover, $(\ell^\infty, \mathcal{F}_{f,F})$ is a differentiable dynamical system. Suppose furthermore, that the local map f satisfies the following chaotic hypothesis:

H1) *There exists a finite collection $\{I_i = [a_i, b_i]\}_{i=1}^p$ of pairwise disjoint bounded and closed intervals, such that for each $1 \leqslant i \leqslant p$ the following assumptions are satisfied*

1. *f is differentiable on I_i with $\inf_{I_i} |f'| > 1$ and $\sup_{I_i} |f'| < +\infty$,*
2. *there exists $1 \leqslant j \leqslant p$ such that $I_j \subset \mathrm{Int}(f(I_i))$,*
3. *$I_j \cap \mathrm{Int}(f(I_i)) \neq \emptyset$ implies $I_j \subset \mathrm{Int}(f(I_i))$.*

In this case, if $\mathcal{C} = \bigcup_{i=1}^p I_i \subset I$, the symbolic dynamics for $f|_{\mathcal{C}}$ is given by the topological Markov chain Ω_A with transition matrix $A = (a_{ij})$, where $a_{ij} = 1$ if and only if $I_j \subset \mathrm{Int}(f(I_i))$. Precisely, the set $\mathcal{C}$ contains an f–invariant set Λ such that $f|_\Lambda$ is topologically conjugate to the shift on Ω_A.

Consider the following extended system: let us call $\Omega_{\mathcal{F}} \doteq \Omega_A^{\mathbb{Z}}$ (the set of all bi–infinite sequences of elements of Ω_A); endow this set with the uniform metric,

$$\Delta(w, w') = \sup_{s\in\mathbb{Z}} \mathrm{d}(w_s, w'_s), \qquad w_s = (w_s^{(0)}, w_s^{(1)}, \ldots) \in \Omega_A$$

where the distance in Ω_A is given by

$$\mathrm{d}(v, v') = \sum_{k=0}^{\infty} \frac{|v^{(k)} - {v'}^{(k)}|}{q^k},$$

for some $q > 1$ fixed. Introduce the dynamics in $\Omega_{\mathcal{F}}$ by using the time–translation operator $(\sigma\, w)_s^{(k)} = w_s^{(k+1)}$.

This yields to the dynamical system $(\Omega_{\mathcal{F}}, \sigma)$ which is called the *spin system* (see [7]). In fact, this system is the direct product of identical Markov chain.

Theorem 2.1 *If f satisfies H1, then there exists $\gamma_{f,F} > 0$ such that, for any coupling parameter $|\gamma| < \gamma_{f,F}$, we can find an $\mathcal{F}_{f,F}$–invariant closed totally disconnected set $\Lambda_{f,F} \subset \mathcal{C}^{\mathbb{Z}}$, so that there exists a homeomorphism $\Pi_{f,F} : \Omega_{\mathcal{F}} \to \Lambda_{f,F}$ satisfying $\mathcal{F}_{f,F} \circ \Pi_{f,F} = \Pi_{f,F} \circ \sigma$. Moreover, $u \in \Lambda_{f,F}$ if and only if $\mathcal{F}^n_{f,F}(u) \in \mathcal{C}^{\mathbb{Z}}$ for all $n \in \mathbb{N}$.*

Proof. See Refs. 1 and 3.

The theorem means that for weak coupling there is a set $(\Lambda_{f,F})$ on which the dynamics is the same (from topological viewpoint) as without coupling.

In this paper we show that for some f and F, the dynamics is the same not only for points in $\Lambda_{f,F}$. Precisely, a trajectory from point $x \notin \Lambda_{f,F}$ will be proved to go to infinity if F and f satisfy the following additional conditions:

H2) *The coupling map F grows no faster than local map f, that is, there exist $\delta > 0$ and $K > 0$ such that for all $y \in \mathbb{R}$*

$$\sup_{|x_j|<|y|} |F(x_1, ..., x_{2r+1})| < \max\{\delta\, |f(y)|,\ K\}$$

and

H3) *There exist $\varepsilon > 0$, $C > 1$ and $M > 0$, such that*

1. *if $x \notin \mathcal{C} = \bigcup_{i=1}^p I_i$, then $|f^n(x)| \to \infty$ as $n \to \infty$.*
2. *if $|x| \geqslant M$, then $|f(x)| \geqslant C|x|$,*
3. $\inf\{|f(x)| \;:\; |f'(x)| \leqslant 1 + \varepsilon\} > M$

This hypothesis says that any trajectory of f, starting from non expanding (and weakly expanding) point, goes to infinity.

The main theorem of the work is

Theorem 2.2 *Consider the map $\mathcal{F}_{f,F}$ given by Equation (3) with f and F satisfying H1, H2, H3 and $|\gamma|$ being small enough. Let $u(n) \doteq \mathcal{F}^n_{f,F}(u_0)$. Then $\|\, u(n) \,\|_\infty \to \infty$ as $n \to \infty$ if and only if $u_0 \notin \Lambda_{f,F}$.*

Before proving the theorem we show that $f(x) = bx(x^2 - 1)$ satisfies the hypothesis H1 and H3 for large enough b.

3. One example of local map

Consider the standard cubic non–linearity $f(x) = bx(x^2-1)$ with $b > 0$ (see Fig. 1) and make $\mathcal{C} = I \setminus A$, where

$$I = \left[-\sqrt{\frac{1+b}{b}}, \sqrt{\frac{1+b}{b}}\right]$$

and

$$A = \left\{ x \in I : |\, f(x) \,| > \sqrt{\frac{1+b}{b}} \right\}.$$

A direct calculation shows that if $b > 3$ then the local extremes of f are, in modulus, bigger than $\sqrt{(1+b)/b}$; so, $\mathcal{C} = I_1 \cup I_2 \cup I_3$ with

$$I_1 = \left[-\sqrt{\frac{1+b}{b}}, \frac{-\sqrt{b^2+b} - \sqrt{b^2-3b}}{2b}\right],$$

$$I_2 = \left[\frac{-\sqrt{b^2+b} + \sqrt{b^2-3b}}{2b}, \frac{\sqrt{b^2+b} - \sqrt{b^2-3b}}{2b}\right],$$

$$I_3 = \left[\frac{\sqrt{b^2+b} + \sqrt{b^2-3b}}{2b}, \sqrt{\frac{1+b}{b}}\right].$$

Moreover, if

$$b > b_0 = \frac{4+6\sqrt{2}}{4}$$

the slope of f in $\mathcal{C}$ is, in modulus, bigger than one; in fact, f satisfies the hypothesis H1 and also satisfies H3 (if $|f'(x)| \leqslant 1$, then $f(x) \notin I$ and, if $x \notin \mathcal{C}$, then $|\, f^n(x) \,| \to \infty$ as $n \to \infty$).

4. Proof of theorem 2.2

By construction $[1, 3]$, $u \in \Lambda_{f,F}$ if and only if $\mathcal{F}^n_{f,F}(u) \in \mathcal{C}^{\mathbb{Z}}$ for all $n \in \mathbb{N}$. So, to prove theorem (2.2) it is enough to prove

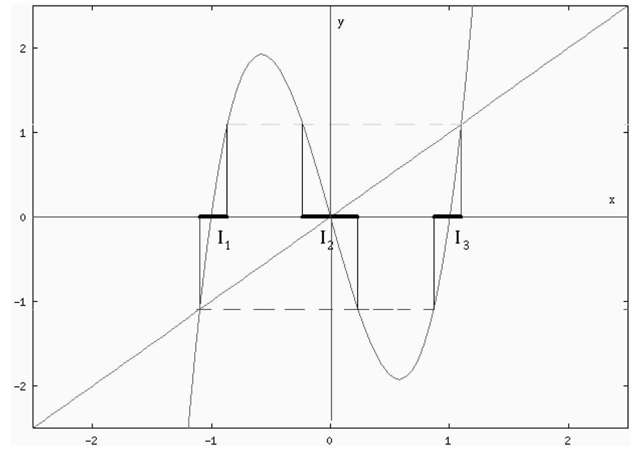

FIGURE 1. The cubic non–linearity $f(x) = bx(x^2 - 1)$.

Theorem 4.1 *Consider the map $\mathcal{F}_{f,F}$ given by Equation (3), with f and F satisfying H1, H2, H3 and $|\gamma| \ll 1$. Let $u(n) \doteq \mathcal{F}^n_{f,F}(u) \notin \mathcal{C}^{\mathbb{Z}}$ for some $n \in \mathbb{N}$. Then $\|\, u(n) \,\|_\infty \to \infty$ as $n \to \infty$.*

In order to prove this theorem, we need some preliminary constructions and propositions.

Let

$$\ell^\infty(\mathbb{Z} \times \mathbb{N}) \doteq \{\, u = \{\, u_{st} \,\}_{s \in \mathbb{Z},\, t \in \mathbb{N}} : \|\, u \,\|_\infty < \infty \},$$

where

$$\|\, u \,\|_\infty = \sup_{s \in \mathbb{Z},\, t \in \mathbb{N}} |u_{st}|.$$

Introduce the map $\mathcal{G}_\gamma : \ell^\infty(\mathbb{Z} \times \mathbb{N}) \to \ell^\infty(\mathbb{Z} \times \mathbb{N})$ defined by

$$(\mathcal{G}_\gamma(u))_{st} = f(u_{st}) + \gamma F(\{u_{st}\}^r) - u_{s,t+1}, \qquad (4)$$

where $\{u_{st}\}^r = (u_{s-r,t}, \ldots, u_{s+r,t})$. The bounded orbits of the original system $(\ell^\infty, \mathcal{F}_{f,F})$, are solutions of the equation

$$(\mathcal{G}_\gamma(u)) = 0, \quad u \in \ell^\infty(\mathbb{Z} \times \mathbb{N}). \qquad (5)$$

Let us write a few words on the strategy to investigate Eq. (5). First, we show that Eq. (5) has no solutions when $\|\, u \,\|_\infty$ is large (lemma 4.2). Second, we study the linear part of Eq. (5) for $\gamma = 0$. Third, we prove theorem 4.1 applying the implicit function theorem.

Lemma 4.2 *Suppose that f and F satisfy H2 and H3. Then, for all*

$$|\gamma| < \min\left\{ \frac{(C-1)}{\delta C}, \ \frac{(C-1)M}{K} \right\}$$

and all $u \in \ell^\infty(\mathbb{Z} \times \mathbb{N})$, $\|u\|_\infty \geqslant M$ implies $\mathcal{G}_\gamma(u) \neq 0$. (here the constants C, K, M and δ are from hypothesis H2 and H3).

Proof: It is enough to make the proof for $\gamma > 0$. Let $\|u\|_\infty = m \geqslant M$, then for any $\varepsilon > 0$ there exists a pair (s_0, t_0) for which $|u_{s_0 t_0}| \geqslant m - \varepsilon$. Moreover, it is clear that

$|u_{st}| \leqslant m$ for all $(s,t) \in \mathbb{Z} \times \mathbb{N}$. Then

$$|(\mathcal{G}_\gamma(u))_{s_0t_0}| = |f(u_{s_0t_0}) - u_{s_0,t_0+1} + \gamma F(\{u_{s_0t_0}\}^r|$$
$$\geqslant |f(u_{s_0t_0})| - |u_{s_0,t_0+1}| - \gamma |F(\{u_{s_0t_0}\}^r|$$
$$\geqslant \min\left\{(1-\gamma\delta)|f(u_{s_0t_0})|,\ |f(u_{s_0t_0})| - \gamma K\right\} - |u_{s_0,t_0+1}|$$
$$\geqslant \min\left\{(1-\gamma\delta)(Cm-\tilde{\varepsilon}),\ Cm-\tilde{\varepsilon}-\gamma K\right\} - m,$$

where $\tilde{\varepsilon} \to 0$ when $\varepsilon \to 0$ due to continuity of f. So, one can find ε and, consequently, (s_0,t_0) such that $|(\mathcal{G}_\gamma(u))_{s_0t_0}| > 0$. We also need some estimates on $\|(D_u\mathcal{G}_0(u))^{-1}\|_\infty$.

Propositon 4.3 *If $|f'(u_{st})| \geqslant 1+\varepsilon$ for all $(s,t) \in \mathbb{Z} \times \mathbb{N}$ then $\|(D_u\mathcal{G}_0(u))^{-1}\|_\infty \leqslant 1/\varepsilon$.*

Proof:

For arbitrary $\xi \in \ell^\infty(\mathbb{Z}\times\mathbb{N})$ one has

$$(D\mathcal{G}_0(u)(\xi))_{st} = f'(u_{st})\,\xi_{st} \ - \ \xi_{s,t+1}\,.$$

Then, for $s \in \mathbb{Z}$ fixed, we can restrict the operator $\mathcal{G}_0$ to the operator $\tilde{\mathcal{G}_0} : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ with the differential

$$(D\tilde{\mathcal{G}_0}(u(s,\cdot))\xi)_t = a_t(s)\,\xi_t - \xi_{t+1}\,,$$

where $a_t(s) = f'(u_{st})$. We are going to exhibit explicitly the operator $L_s = (D\tilde{\mathcal{G}_0}(u(s,\cdot)))^{-1}$, which is supposed to have the form

$$L_s(\xi)_i = \sum_{j\in\mathbb{N}} c_{ij}(s)\,\xi_j.$$

A direct calculation shows that for the coefficients

$$c_{ij}(s) = \begin{cases} 0 & \text{if} \quad j < i \\ \displaystyle\prod_{t=i}^{j}(a_t(s))^{-1} & \text{if} \quad j \geqslant i \end{cases},$$

L_s (if bounded) is the inverse operator of $D\tilde{\mathcal{G}_0}(u(s,\cdot))$.

It is simple to show that

$$\|\,L_s\,\|_\infty = \sup_{i\in\mathbb{N}}\left\{\sum_{j\in\mathbb{N}}|\,c_{ij}(s)\,|\right\}.$$

The conclusion follows from the inequality

$$\sum_{j\in\mathbb{N}}|\,c_{ij}(s)\,| = \sum_{j\geqslant i}|\,c_{ij}(s)\,| \leqslant \sum_{j=1}^{\infty}\left(\frac{1}{1+\varepsilon}\right)^j = \frac{1}{\varepsilon}.$$

which holds for any $i \in \mathbb{N}$, and from the fact that $\|(D_u\mathcal{G}_0(u))^{-1}\|_\infty = \sup_{s\in\mathbb{Z}}\|L_s\|_\infty$.

Finally, we need

Proposition 4.4 *Let $\ell^\infty(\mathbb{Z}\times\mathbb{N}) \supset S = \{u\ :\ \mathcal{G}_0(u) = 0\}$. Then, for all $x \in S$ and $\varepsilon > 0$ small enough, there exists $\gamma' > 0$ such that if $|\gamma| < \gamma'$, the equation $\mathcal{G}_\gamma(u) = 0$ has a unique solution u^* in the open ball $O_\varepsilon(x)$.*

Proof: The existence of such γ' and ε for any fixed $x \in S$ immediately follows from the Implicit Function Theorem. So, in order to prove the proposition one should check that all estimates required are uniform for $x \in S$.

In order to prove theorem 4.1 we need the following simple consequence of the implicit function theorem.

Lemma 4.5 *Let B be a Banach space, let*

$$\mathcal{G}(\cdot,\cdot) : \mathbb{R}\ \times\ B\ \to\ B$$

be a C^1 map and $X \subset B$ be a closed region. Suppose that $\mathcal{G}(0,x) \neq 0$ for all $x \in X$, but there exist x_1 and $\gamma_1 > 0$ such that $\mathcal{G}(\gamma_1,x_1) = 0$. Then there exist $x_0 \in X$ and $0 < \gamma_0 \leqslant \gamma_1$ such that either

1. $\mathcal{G}(\gamma_0,x_0) = 0$ *and* $x_0 \in \partial X$

or

2. $D_x\mathcal{G}(\gamma_0,x_0)$ *does not have bounded inverse operator.*

Proof of theorem 4.1.

It is enough again to consider $\gamma > 0$. By Lemma 4.2 we can find $M > 0$ and $\gamma_1 > 0$, such that for $\gamma < \gamma_1$ there is no root u of (5) satisfying $\|u\| \geqslant M$. Let $S = \{u\ :\ \mathcal{G}_0(u) = 0\}$. By proposition (4.4) we can take such a small ε, that there is one and only one solution of (5) in $O_\varepsilon(x)$ for $x \in S$ and γ small enough. We only should check that there is no roots in

$$X = \mathrm{Clos}\left(O_M(0)\ \setminus\ \bigcup_{x\in S} O_\varepsilon(x)\right).$$

For all small enough γ there exist no roots on the boundary of X and there exist no roots in X for $\gamma = 0$. Suppose that there exists some root in X for γ small enough. Then, by lemma 4.5, there exist $x \in X$ and $0 < \gamma_0 \leqslant \gamma$, such that $\mathcal{G}_{\gamma_0}(x) = 0$ and $D_x\mathcal{G}_{\gamma_0}(x)$ does not have bounded inverse operator. Since $D_x\mathcal{G}_{\gamma_0}(x)$ is close to $D_x\mathcal{G}_0(x)$, uniformly for all $x \in X$, we have that $\|D_x\mathcal{G}_0(x)\|_\infty$ is large. Then, by proposition 4.3, $|f'(u_{st})| < 1+\varepsilon$ for some (s,t) and small $\varepsilon > 0$. Moreover, the smaller γ_0 is the smaller ε is. So, decreasing, if necessary, the interval for γ, we can make this ε small enough as is required in the hypothesis H3. But then $|u_{st}| > M$ for small enough γ and $u \notin X$.

5. Concluding remarks

Remark 5.1 *If the hypothesis H1 and H3 holds, the symbolic description of the dynamics of $(\Lambda_{f,F},\mathcal{F}_{f,F})$ has to be a Bernoulli shift.*

First of all, we can choose, increasing b_i and decreasing a_i, the intervals $I_i = [a_i,b_i]$ in such a way that either

$$|f'(a_i)| = 1+\varepsilon$$

or $a_i = -M$ and either

$$|f'(b_i)| = 1+\varepsilon$$

or $b_i = M$. So, $|f(a_i)|$ and $|f(b_i)|$ are greater than M. We claim that either $f(a_i) < -M$ and $f(b_i) > M$ or $f(b_i) < -M$ and $f(a_i) > M$. (If not, we have for example,

$$M < f(a_i) \leqslant f(b_i).$$

But from hypothesis H1, it follows that

$$f([a_i, b_i]) \bigcap [-M, M] \neq \emptyset.$$

So, we could find points $\alpha_i, \beta_i \in I_i$ such that $f(\alpha_i) = f(\beta_i)$. By the Mean Value Theorem, $f'(\gamma_i) = 0$ for some $\gamma_i \in I_i$ which contradicts H1.1.

Remark 5.2 *We can easily estimate how small the coupling between the chaotic subsystems in theorem 2.2 should be under the additional assumption, namely,*

$$\nu \doteq \min_{x \in [-M,M] \setminus \mathcal{C}} |f(x)| - M > 0.$$

In this case the proof of theorem 2.2 is much more simple.

Let us call,

$$\alpha \doteq \max_{1 \leqslant s \leqslant 2r+1} |F(y_1, \ldots, y_{2r+1})|$$

and

$$\beta \doteq \max_{1 \leqslant s \leqslant 2r+1} \left| \frac{\partial F(y_1, \ldots, y_{2r+1})}{\partial x_s} \right|,$$

where the maximums are taken over all the possible choices of $y_s \in [-M, M]$, $(1 \leqslant s \leqslant 2r + 1)$.

1. *In order to satisfy theorem 2.1 we make*

$$\gamma_1 \leqslant \left\{ \frac{\mu}{2\alpha}, \frac{\varepsilon}{\beta(2r+1)} \right\},$$

where $\mu = \min\{\min f|_{\mathcal{C}} - M,\ M - \max f|_{\mathcal{C}}\}$ *and* M *is the constant in hypothesis H3 (see [1]).*

2. *In order to satisfy lemma 4.2 we make*

$$\gamma_2 \leqslant \min\left\{ \frac{(C-1)}{\delta C},\ \frac{(C-1)M}{K} \right\},$$

where C, K, M *and* δ *are again the constants from hypothesis H2 and H3.*

3. *Finally, in order that all points in* $[-M, M] \setminus \mathcal{C}$ *maps out of interval* $[-M, M]$, *we take*

$$\gamma_3 \leqslant \frac{\nu}{\alpha},$$

where

$$\nu = \min_{x \in [-M,M] \setminus \mathcal{C}} |f(x)| - M > 0.$$

So, if

$$|\gamma| \leqslant \min\{\gamma_1, \gamma_2, \gamma_3\}$$

then conclusions of theorems 2.1 and theorem 2.2 hold.

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