On the figure eight orbit of the three-body problem

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A new solution to the three-body problem interacting through gravitational forces with equal masses and zero angular momentum, has been recently discovered. This is a periodic symmetric orbit where the particles follow a figure eight trajectory in the plane. They alternate between six isosceles-aligned positions and six isosceles triangle positions in a periodic orbit composed by twelve equivalent segments. The condition of zero angular momentum is considered assuming that the three masses can be equal or different, yielding in both cases the same final expression for the kinetic energy. We found that the property of this orbit of having isosceles configurations, is a general feature to be found in any orbit of the equal-mass case, associated with an increase of $\pi/6$ in one angle of our set of coordinates. The figure-eight solution is determined by expanding two of our coordinates in a Fourier series of that angle, by using the Jacobi-Maupertuis principle as opposed to the standard Lagrangian action. The time and the angle conjugated to the angular momentum are also expressed in terms of that same angle.

Keywords: Three-body problem; zero angular momentum; equal-mass case; figure-eight orbit; Jacobi's action.

Recientemente se descubrió una solución nueva del problema de tres cuerpos que interaccionan mediante fuerzas gravitacionales entre masas iguales y con momento angular cero. Se trata de una órbita simétrica periódica, en la cual las partículas siguen la misma trayectoria con forma de ocho en el plano. Hay una alternancia entre seis posiciones isósceles alineadas y seis posiciones triangulares isósceles en la órbita, compuesta por doce segmentos equivalentes. La condición de momento angular cero se considera con el supuesto de que las tres masas pueden ser iguales o diferentes, dando lugar en ambos casos a la misma expresión final para la energía cinética. Encontramos que la propiedad de esta órbita de tener configuraciones isósceles, es una característica general que se encuentra en cualquier órbita del caso de masas iguales, asociada con un incremento de $\pi/6$ en un ángulo de nuestro conjunto de coordenadas. La trayectoria con forma de ocho se obtiene mediante la expresión de dos de nuestras coordenadas como una serie de Fourier de dicho ángulo, haciendo uso del principio de Jacobi-Maupertuis en lugar de la acción estándar de Lagrange. El tiempo y el ángulo conjugado al momento angular se encuentran también en términos del mismo ángulo.

Descriptores: Problema de tres cuerpos; momento angular nulo; caso de masas iguales; órbita con forma de ocho; forma de Jacobi del principio de Maupertuis.

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1. Introduction

A new solution to the three-body problem, interacting by gravitational forces for three equal masses, was found recently by Moore [1] and analized by Chenciner and Montgomery [2], that increases the number of the classical particular solutions discovered by Euler in 1765 and Lagrange in 1772, and focuses the interest of many researchers to look for new particular cases of this motion.

The study in this paper was made in the coordinate system of Piña and Jiménez [3-5], which was introduced recently for considering the general three-body problem with three different masses m_1, m_2, m_3 interacting through gravitational forces. A brief summary of the most important features and results follows. This coordinate system, with origin at the center of mass, uses as dynamic coordinates the three Euler angles ϕ, θ, ψ that determine the position of the plane of the particles and the orientation of the principal inertia directions in this plane. The three other coordinates are two distances R_1, R_2 related to the two independent moments of inertia and one auxiliary angle σ. These last three coordinates are functions only of the three masses and the three distances between the particles.

The Lagrangian function to derive the equations of motion in the new coordinates is the difference of the kinetic K and potential V energies

$$\mathcal{L} = K - V; \tag{1}$$

the kinetic energy is given by

$$K = \mu\left[\frac{1}{2}\dot{R_1}^2 + \frac{1}{2}\dot{R_2}^2 + \frac{1}{2}(R_1^2+R_2^2)\dot{\sigma}^2 - 2\,R_1\,R_2\dot{\sigma}\omega_3 + \frac{R_1^2}{2}\omega_1^2 + \frac{R_2^2}{2}\omega_2^2 + \frac{R_1^2+R_2^2}{2}\omega_3^2\right], \tag{2}$$

where μ is the reduced mass

$$\mu = \sqrt{\frac{m_1\,m_2\,m_3}{m_1+m_2+m_3}}, \tag{3}$$

the upper dot means time derivative; ω_1, ω_2, ω_3, are the components of the angular velocity vector in Euler angles [6]

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \dot{\phi}\begin{pmatrix}\sin\theta\sin\psi\\ \sin\theta\cos\psi\\ \cos\theta\end{pmatrix} + \dot{\theta}\begin{pmatrix}\cos\psi\\ -\sin\psi\\ 0\end{pmatrix} + \dot{\psi}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}, \tag{4}$$

and the potential energy is

$$V = -\frac{Gm_2m_3}{r_{23}} - \frac{Gm_3m_1}{r_{31}} - \frac{Gm_1m_2}{r_{12}}, \qquad (5)$$

where G is the gravitational constant. The distance between particles r_{23}, r_{31}, r_{12}, in terms of the new coordinates can be written from

$$\begin{pmatrix} r_{23}^2 \\ r_{31}^2 \\ r_{12}^2 \end{pmatrix} = \mathbf{B} \begin{pmatrix} R_1^2 \sin^2\sigma + R_2^2 \cos^2\sigma \\ R_1^2 \cos^2\sigma + R_2^2 \sin^2\sigma \\ (R_2^2 - R_1^2) 2 \sin\sigma \cos\sigma \end{pmatrix}, \qquad (6)$$

where $\mathbf{B}$ is a square, constant matrix whose elements depend only on the masses (see the previous papers, Refs. [3-5] for more details).

2. Zero angular momentum

It is well known that the equations of motion are deduced from the Hamilton variational principle by minimizing the action

$$\delta \int_{t1}^{t2} \mathrm{d}t (K - V) = 0, \qquad (7)$$

over different trajectories with the same end points.

The equations have several constants of motion. Besides the conservation of linear momentum that has been accounted for by choosing the origin of position vectors at the center of mass and assuming that the center of mass is at rest, we also have the energy constant

$$E = K + V, \qquad (8)$$

and the conservation of the angular momentum vector [5], which can be written as

$$\mathbf{L} = \mathcal{G} \left(R_1^2 \omega_1, R_2^2 \omega_2, (R_1^2 + R_2^2)\omega_3 - 2\, R_1\, R_2 \dot\sigma \right)^{\mathrm{T}}, \qquad (9)$$

where $\mathcal{G}$ is the rotation matrix in Euler angles. This matrix transforms from the frame determined by the three principal inertia directions to the inertial frame. Its form in terms of the Euler angles is selected with the same definition as used in many texts of dynamics [6]. A more explicit definition, including a figure, was enclosed in our Ref. [3].

The focus of interest in this paper is on the three-body problem when the angular momentum vector is the null vector. In this case, the first two components of the angular velocity vector must be zero,

$$\omega_1 = \omega_2 = 0, \qquad (10)$$

which implies that the motion occurs in a constant plane. The Euler rotations by angles θ and ϕ are then supressed. As a consequence, it is sufficient to take into account just the rotation angle ψ, instead of three Euler angles, in order to transform from the inertial reference frame to the frame of principal axes of inertia.

As a function of the new coordinates, the kinetic energy becomes

$$K = \mu \left[\frac{1}{2} \dot{R_1}^2 + \frac{1}{2} \dot{R_2}^2 + \frac{1}{2} (R_1^2 + R_2^2) \dot\sigma^2 - 2\, R_1\, R_2 \dot\sigma \dot\psi + \frac{R_1^2 + R_2^2}{2} \dot\psi^2 \right]. \qquad (11)$$

Conservation of angular momentum in the plane is expressed by the fact that the ψ coordinate is a cyclic variable, and its conjugate momentum is therefore a constant which from now on is actually equal to zero

$$p_\psi = \frac{\partial K}{\partial \dot\psi} = -2\, R_1\, R_2 \dot\sigma + (R_1^2 + R_2^2) \dot\psi = 0. \qquad (12)$$

In similar cases in which cyclic variables occur, it is convenient to follow the Routhian formalism, in which a new Lagrangian function $\mathcal{R}$ is defined as

$$\mathcal{R}(R_1, R_2, \sigma, \dot R_1, \dot R_2, \dot\sigma, p_\psi) = K - V - \dot\psi p_\psi \qquad (13)$$

and uses p_ψ as a constant parameter in the Euler-Lagrange variational equations. But in this case of angular momentum equal to zero, $\mathcal{R}$ simplifies to

$$\mathcal{R} = \frac{\mu}{2} \left(\dot{R_1}^2 + \dot{R_2}^2 + \frac{(R_2^2 - R_1^2)^2}{R_1^2 + R_2^2} \dot\sigma^2 \right) - V, \qquad (14)$$

which is of the form of an usual Lagrangian in the three coordinates (R_1, R_2, σ), quadratic in the velocities, and with a kinetic energy metric in orthogonal coordinates.

3. The equal mass case

For three equal masses, the Euler angles connecting the inertial reference frame and the principal moments of inertia axes are again used. Contrasting with the previous section, all of the relations in this section are valid for arbitrary angular momentum.

However, in the three equal mass case

$$\mathbf{m} = m_1 = m_2 = m_3, \qquad (15)$$

the vectors $\mathbf{a}$ and $\mathbf{b}$ previously given in papers [3-5] are not well defined and new definitions are necessary.

Since the vector formed with the masses

$$\mathbf{m} = (m_1, m_2, m_3)$$

is now proportional to the $(1, 1, 1)$ vector, the center of mass equations are just

$$s_{1x} + s_{2x} + s_{3x} = 0, \quad s_{1y} + s_{2y} + s_{3y} = 0, \qquad (16)$$

where sjx, sjy, sjz are the position components of the three particles in the frame of the principal directions of inertia.

Two unit vectors $\mathbf{a}$ and $\mathbf{b}$ exist, which are orthogonal to the mass vector and orthogonal to each other

$$\mathbf{a}\cdot\mathbf{m} = \mathbf{b}\cdot\mathbf{m} = \mathbf{a}\cdot\mathbf{b} = 0, \tag{17}$$

that may be taken for convenience as

$$\mathbf{a} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \tag{18}$$

$$\mathbf{b} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right). \tag{19}$$

The particle coordinates in the frame of the principal axes of inertia are then written as linear combinations of these two vectors

$$(s_{1x}, s_{2x}, s_{3x}) = \alpha\mathbf{a} + \beta\mathbf{b} \tag{20}$$

and

$$(s_{1y}, s_{2y}, s_{3y}) = \gamma\mathbf{a} + \delta\mathbf{b}\,. \tag{21}$$

The frame of the principal axes of inertia gives the additional restriction

$$s_{1x}s_{1y} + s_{2x}s_{2y} + s_{3x}s_{3y} = 0, \tag{22}$$

which imposes the condition that

$$\alpha\gamma + \beta\delta = 0\,. \tag{23}$$

In terms of the coefficients of Eqs. (20) and (21), the moments of inertia become

$$I_2 = \sum_j m_j s_{jx}^2 = m(\alpha^2 + \beta^2) \tag{24}$$

and

$$I_1 = \sum_j m_j s_{jy}^2 = m(\gamma^2 + \delta^2)\,. \tag{25}$$

The coefficients $(\alpha, \beta, \gamma, \delta)$ may also be expressed in the same form we have used in previous publications [3-5]

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} R_2 & 0 \\ 0 & R_1 \end{pmatrix}\begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} = \begin{pmatrix} R_2\cos\sigma & R_2\sin\sigma \\ -R_1\sin\sigma & R_1\cos\sigma \end{pmatrix}, \tag{26}$$

and the principal moments of inertia are written in terms of the distances R_1 and R_2 simply as

$$I_1 = mR_1^2\,, \quad I_2 = mR_2^2\,. \tag{27}$$

Explicitly, one has

$$\begin{pmatrix} s_{1x} & s_{2x} & s_{3x} \\ s_{1y} & s_{2y} & s_{3y} \end{pmatrix} = \begin{pmatrix} \alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{\sqrt{6}} & -\alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{\sqrt{6}} & -\beta\frac{2}{\sqrt{6}} \\ \gamma\frac{1}{\sqrt{2}} + \delta\frac{1}{\sqrt{6}} & -\gamma\frac{1}{\sqrt{2}} + \delta\frac{1}{\sqrt{6}} & -\delta\frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} R_2 & 0 \\ 0 & R_1 \end{pmatrix}\begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} = -\sqrt{\frac{2}{3}}\begin{pmatrix} R_2 & 0 \\ 0 & R_1 \end{pmatrix}\begin{pmatrix} \sin(\sigma - 2\pi/3) & \sin(\sigma + 2\pi/3) & \sin\sigma \\ \cos(\sigma - 2\pi/3) & \cos(\sigma + 2\pi/3) & \cos\sigma \end{pmatrix}. \tag{28}$$

From these expresions, we compute the square of the distances of the particles in terms of the new coordinates as

$$\begin{pmatrix} r_{23}^2 \\ r_{31}^2 \\ r_{12}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & -\sqrt{3} \\ \frac{1}{2} & \frac{3}{2} & \sqrt{3} \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} R_2^2\cos^2\sigma + R_1^2\sin^2\sigma \\ R_2^2\sin^2\sigma + R_1^2\cos^2\sigma \\ (R_2^2 - R_1^2)\sin\sigma\cos\sigma \end{pmatrix}. \tag{29}$$

We have previously found [3-5] this form [cf. Eq. (6)] but the constant matrix is much simpler in the equal mass case. Computing the matrix product in (29), and rearranging in a way that will show useful in the next pages, the right hand side can also be expressed in terms of the angle 2σ in the form

$$\begin{pmatrix} r_{23}^2 \\ r_{31}^2 \\ r_{12}^2 \end{pmatrix} = (R_1^2 + R_2^2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (R_2^2 - R_1^2)\begin{pmatrix} \cos(2\sigma + \frac{2\pi}{3}) \\ \cos(2\sigma - \frac{2\pi}{3}) \\ \cos(2\sigma) \end{pmatrix}. \tag{30}$$

The purely kinematic Eq. (30) implies an important result. If the angle σ is a multiple of $\pi/6$, then two of the distances are of the same magnitude, forming either an isosceles triangle or an isosceles collinear configuration. Assuming that σ increases monotonically with the dynamical evolution of the system, whenever the angle σ has one of these values, two of the sides are equal. The sequence of the pair of equal distances having a particle at a common vertex is permuted cyclically as shown in the following table, where the values of the squares of the distances of the particles are expressed

TABLE I. Values of the squares of the inter-particle distances for the values of the angle σ when two sides are of the same length.

σ	r_{23}^2	r_{31}^2	r_{12}^2
$0, 6\pi/6$	$\frac{3}{2}R_1^2+\frac{1}{2}R_2^2$	$\frac{3}{2}R_1^2+\frac{1}{2}R_2^2$	$2R_2^2$
$\pi/6, 7\pi/6$	$2R_1^2$	$\frac{1}{2}R_1^2+\frac{3}{2}R_2^2$	$\frac{1}{2}R_1^2+\frac{3}{2}R_2^2$
$2\pi/6, 8\pi/6$	$\frac{3}{2}R_1^2+\frac{1}{2}R_2^2$	$2R_2^2$	$\frac{3}{2}R_1^2+\frac{1}{2}R_2^2$
$3\pi/6, 9\pi/6$	$\frac{1}{2}R_1^2+\frac{3}{2}R_2^2$	$\frac{1}{2}R_1^2+\frac{3}{2}R_2^2$	$2R_1^2$
$4\pi/6, 10\pi/6$	$2R_2^2$	$\frac{3}{2}R_1^2+\frac{1}{2}R_2^2$	$\frac{3}{2}R_1^2+\frac{1}{2}R_2^2$
$5\pi/6, 11\pi/6$	$\frac{1}{2}R_1^2+\frac{3}{2}R_2^2$	$2R_1^2$	$\frac{1}{2}R_1^2+\frac{3}{2}R_2^2$

in terms of our coordinates for values of σ equal to a multiple of $\pi/6$. This is a necessary and sufficient condition for two of the sides to be equal. The distances repeat their expressions after σ increases by π.

These results are independent of the constant values of the energy and of the angular momentum vector. They would still hold even if the potential energy were different, as long as it is only a function of the inter-particle distances. The only relevant restriction is the equality of the three masses.

The Routhian formulation for zero angular momentum can be employed with minor modifications in the three equal mass case. Mass μ is replaced by the common value m of the masses except for an irrelevant different factor of $\sqrt{3}$ in the coordinates R_1^2 and R_2^2. Additionally, the potential energy assumes the simplest possible expression as a consequence of the increased mass symmetry. Further results for the zero angular momentum and equal mass case are obtained in the following section.

4. The figure-eight orbit

In this section we study the figure-eight orbit for three equal masses, discovered by Moore [1] and described by Chenciner and Montgomery [2], which is a zero angular momentum solution. It has remarkable symmetry properties that are here described very simply in terms of our coordinates. Other forms of representing the symmetries of this orbit have been used by Chenciner and Montgomery [2] and Marchal [7].

These symmetries result from the following properties.

1. A cyclic permutation of the particles is equivalent to a shift in the σ value by an angle $2\pi/3$, as follows from the property

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}, \quad (31)$$

where the components of the basis vectors $\mathbf{a}$ and $\mathbf{b}$, grouped in a matrix as in Eq. (28), are left-transformed or right-transformed with equivalent result.

2. In a similar way, permutation of particles 1 and 2 is equivalent to a change of sign of vector $\mathbf{a}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}. \quad (32)$$

But this is also equivalent to a change of sign of coordinates σ and R_2 as follows from

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{pmatrix}. \quad (33)$$

Denote by $\mathbf{X}(t)$ the coordinates of the three particles in the inertial system,

$$\mathbf{X}(t) = \mathcal{G} \begin{pmatrix} s_{1x} & s_{2x} & s_{3x} \\ s_{1y} & s_{2y} & s_{3y} \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \times \begin{pmatrix} s_{1x} & s_{2x} & s_{3x} \\ s_{1y} & s_{2y} & s_{3y} \end{pmatrix}. \quad (34)$$

Calling T the period of the orbit, one of the symmetries of the orbit is expressed in matrix notation as

$$\mathbf{X}(t+T/6) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X}(t) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (35)$$

This symmetry means that when time is increased by one-sixth of the period $(t \rightarrow t+T/6)$, the particle coordinates are permuted cyclically $(1\rightarrow 3\rightarrow 2\rightarrow 1)$, with a change of sign in the horizontal coordinate along the symmetry axis intersecting three points of the orbit. In terms of our coordinates, this symmetry implies an increase of σ by $\pi/3$ and a change of sign of the ψ and R_1 coordinates, while the R_2 coordinate repeats its value (since it is periodic in time with period $T/6$). The square of the transformations representing this symmetry implies just a cyclic permutation of the particles when time is increased by $T/3$ and simultaneously σ increases its value by $2\pi/3$.

The orbit has another symmetry, expressing time reversal invariance, namely

$$\mathbf{X}(-t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{X}(t) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (36)$$

which expresses the permutation of particles 1 and 2, with a change of sign of the two coordinates of the three particles. This is equivalent in our coordinates to a change of sign of σ and R_1. These quantities are therefore odd functions of time, and R_1 should also be an odd function of σ. On the other hand, the coordinates ψ and R_2 are even functions of time and even functions of σ.

Combination of these last two symmetries implies that the orbit can be considered to be formed by twelve equivalent segments. If the orbit is parametrized by σ, this requires that R_2^2 and R_1^2 are functions of $\cos 6\sigma$. Summing up the properties of R_1 and R_2, we put forward that they can be satisfied by Fourier series of the forms

$$R_1(\sigma) = \sin(3\sigma)\sum_{j=0}^{\infty} b_j \cos(6j\sigma), \qquad R_2(\sigma) = \sum_{j=0}^{\infty} c_j \cos(6j\sigma), \tag{37}$$

where b_* and c_* are constant Fourier coefficients.

As a consequence, the determination of the trajectory can be simplified eliminating the time by using Jacobi's form [8] of Maupertuis' principle, which gives

$$0 = \delta \int_0^{\pi/6} \sqrt{2(E-V)\left((\mathrm{d}R_1)^2 + (\mathrm{d}R_2)^2 + \frac{(R_2^2 - R_1^2)^2}{R_1^2 + R_2^2}(\mathrm{d}\sigma)^2\right)}. \tag{38}$$

The formulation in terms of this variational principle, instead of Hamilton's principle (7), has the advantage of reducing the number of dependent coordinates from four to two.

Substitution of the Fourier series (37) for $R_1(\sigma)$ and $R_2(\sigma)$ in the integral of this variational problem gives Jacobi's action as a function of the parameters b_* and c_*. These parameters are then varied in an iterative procedure that reduces the value of the action until no further decrease can be obtained. In this form an aproximation to the equation of the orbit is found.

The angle $\psi(\sigma)$ can be obtained from the angular momentum conservation [Eq. (12)] as

$$\psi(\sigma) = \int_{\pi/6}^{\sigma} \mathrm{d}\sigma \frac{2R_1R_2}{R_1^2 + R_2^2}\,. \tag{39}$$

In particular, the slope angle between the isosceles collinear configuration and the symmetry axis (of the orbit and of the isosceles triangular configuration) is given by the integral

$$-\psi(0) = \int_0^{\pi/6} \mathrm{d}\sigma \frac{2R_1R_2}{R_1^2 + R_2^2}. \tag{40}$$

The relation between the σ coordinate and time follows from energy conservation

$$t(\sigma) = \int_0^{\sigma} \mathrm{d}\sigma \sqrt{\frac{\left(\frac{\mathrm{d}R_1}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}R_2}{\mathrm{d}\sigma}\right)^2 + \frac{(R_2^2-R_1^2)^2}{R_1^2+R_2^2}}{2(E-V(\sigma))}}. \tag{41}$$

The period of the orbit can thus be computed from

$$T = 12 \times \int_0^{\pi/6} \mathrm{d}\sigma \sqrt{\frac{\left(\frac{\mathrm{d}R_1}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}R_2}{\mathrm{d}\sigma}\right)^2 + \frac{(R_2^2-R_1^2)^2}{R_1^2+R_2^2}}{2(E-V(\sigma))}}. \tag{42}$$

5. Numerical results

We have been able to reproduce with some accuracy C. Simó's [2,9] and D. Viswanath's [10] numerical computations for the figure eight orbit, finding the first Fourier coefficients which minimize the integral in (38). Our values are reported in the following table. All our computations, made in double precision, used the same physical units reported in the first page of Chenciner and Montgomery's publication [2]. We have used the energy value $E = -1.28714199563186$, consistent with the extremely precise computations of Simó and Viswanath. The initial value $R_2(0) = \sqrt{2}$, fixes the scale of our physical units with $m_1 = m_2 = m_3 = G = 1$. For $dR_1(0)/d\sigma$, instead of using Simó's and Viswanath's results, we chose a value 2.17039427473812, determined by a compromise between minimization of the Jacobi integral and fulfillement of the condition

$$I = \int_0^{\pi/6} \mathrm{d}\sigma \sqrt{2(E-V)\left[\left(\frac{\mathrm{d}R_1}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}R_2}{\mathrm{d}\sigma}\right)^2 + \frac{(R_2^2-R_1^2)^2}{R_1^2+R_2^2}\right]} = -E\,T/6\,, \tag{43}$$

within double precision.

This equation is a direct consequence of the virial theorem, written in terms of the σ coordinate and should be satisfied by the exact solution. Transforming this integral to the time integration variable we obtain the value of the action predicted by the virial theorem

$$\oint \mathrm{d}t(K - V) = -3ET\,. \tag{44}$$

This result is compatible with the value published by Chenciner and Montgomery [2] if one uses Viswanath's [10] physical units.

These values predict Simó's orbit and the values $I = 1.357058258032642$, $T = 6.32591398293920$ and $-\psi(0) = 0.245547563748942$, which are close to the reported numbers.

TABLE II. Fourier coefficients of the $R_1(\sigma)$ and $R_2(\sigma)$ coordinates as they appear in Eqs. (37)

j	b_j	c_j
0	$5.83253354692453 \times 10^{-1}$	$1.36151499375215 \times 10^{+0}$
1	$1.12255306874746 \times 10^{-1}$	$4.34551784798245 \times 10^{-2}$
2	$2.12208094325769 \times 10^{-2}$	$7.15968013904149 \times 10^{-3}$
3	$4.95111334363167 \times 10^{-3}$	$1.55379542791284 \times 10^{-3}$
4	$1.28442417181987 \times 10^{-3}$	$3.85163088496634 \times 10^{-4}$
5	$3.55233980418980 \times 10^{-4}$	$1.03300085959211 \times 10^{-4}$
6	$1.02379675997835 \times 10^{-4}$	$2.91885621732155 \times 10^{-5}$
7	$3.02059195873733 \times 10^{-5}$	$8.56013124696136 \times 10^{-6}$
8	$8.90024768111640 \times 10^{-6}$	$2.58177637808029 \times 10^{-6}$
9	$2.47208813196311 \times 10^{-6}$	$7.96722966662622 \times 10^{-7}$
10	$5.37481455254816 \times 10^{-7}$	$2.48770592821083 \times 10^{-7}$
11	$1.53809898323867 \times 10^{-8}$	$7.19630834119502 \times 10^{-8}$
12	$-6.24885009256139 \times 10^{-9}$	$2.84804853491605 \times 10^{-9}$
13	$1.10979730943712 \times 10^{-8}$	$3.78721637622082 \times 10^{-10}$
14	$7.43152755167694 \times 10^{-11}$	$1.84190507506145 \times 10^{-10}$
15	$1.39383449382033 \times 10^{-11}$	$5.78053531587408 \times 10^{-11}$
16	$1.91820131557633 \times 10^{-11}$	$4.50774814987822 \times 10^{-12}$

6. Relation between our coordinates and other coordinates

There are simple relations between our coordinates and the coordinates that have been used by other authors in the equal-mass case.

Define polar-like coordinates (with an extra one-half factor) for the R_1 and R_2 variables and for σ, one-half of the θ angle

$$R_1 = r\cos\phi/2, \quad R_2 = r\sin\phi/2, \quad \sigma = \theta/2. \tag{45}$$

Then the metric associated with the kinetic energy for the zero angular momentum case becomes the Chenciner-Montgomery spherical-like expression [2]

$$ds^2 = dr^2 + \frac{r^2}{4}(\cos^2\phi\, d\theta^2 + d\phi^2). \tag{46}$$

However, in this paper, we found that this relation holds even for three unequal masses.

In addition, our Eq. (30) relating the sides of the triangle to the coordinates, becomes

$$\begin{pmatrix} r_{23}^2 \\ r_{31}^2 \\ r_{12}^2 \end{pmatrix} = r^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r^2\cos^2\phi \begin{pmatrix} \cos(\theta + \frac{2\pi}{3}) \\ \cos(\theta - \frac{2\pi}{3}) \\ \cos\theta \end{pmatrix}, \tag{47}$$

that coincides with Hsiang's Eqs. [11], quoted by Chenciner and Montgomery [2], but which is valid only in the equal-mass case.

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