*Atmósfera (1988)*, **1**, pp. 45-62

# The meridional scale of baroclinic waves with latent heat release

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(Manuscript received August 12, 1987; accepted November 24, 1987)


## RESUMEN

Se desarrolla una teoría de ondas baroclínicas en dos niveles, con humedad, cuasi-geostróficas y con variación meridional, pero sin efecto $\beta$. La formulación es similar a la de Tang y Fichtl (1983) excepto que se permiten variaciones meridionales. Los parámetros básicos son el número rotacional de Froude $F = 2f^2[S_d p_2^2(k_d^2 + \ell^2)]^{-1}$ (donde $f$ es el parámetro de Coriolis, $S_d$ la estabilidad estática en la porción descendente de la onda, $p_2$ la presión en el nivel medio, $k_d$ el número de onda zonal en la parte descendente de la onda, $\ell(=\pi/D)$ el número de onda meridional y $D$ es la extensión meridional de la onda) y un parámetro de calentamiento $\varepsilon$ que es proporcional al gradiente vertical en el nivel central del flujo medio de la razón de mezcla de saturación. Para $\varepsilon \neq 0$ las perturbaciones se caracterizan por una longitud zonal desigual de la porción ascendente o húmeda de la onda ($a$) con respecto a la longitud zonal de la porción descendente o seca de la onda ($b$). El primer modo tiene una pequeña región de fuerte movimiento ascendente y una gran región de débil movimiento descendente ($a/b < 1$), sucediendo lo contrario para el segundo modo ($a/b > 1$). Estas características son similares a las obtenidas por Tang y Fichtl (1983). En el presente trabajo se deduce una ecuación a escala meridional, mostrando tres posibilidades: (i) $\ell/k_d = 0$ (ondas baroclínicas sin variaciones meridionales, discutido en Tang y Fichtl, 1983), (ii) $\varepsilon = 0$ (modelo seco, discutido en Phillips, 1954 sin el efecto $\beta$, y con $\ell$ arbitraria); y (iii) una ecuación bicuadrática en $\ell/k_d$.

La última ecuación contiene esencialmente la información de la influencia de la liberación de calor latente en la escala meridional de las ondas baroclínicas. El modelo de ondas baroclíncias meridionalmente uniformes ($\ell/k_d = 0$) y el modelo seco ($\varepsilon = 0$) son singulares en tanto que sus características no se pueden deducir haciendo $\ell/k_d \to 0$ ó $\varepsilon \to 0$ en esta ecuación bicuadrática. Para $\varepsilon < 0.464$, la razón de la extensión meridional $D$ del dominio zonal, $a + b$, es menor que 0.7. Para $\varepsilon$ y $F$ dadas, esta razón es mayor para el primer modo que para el segundo. La razón de crecimiento en la región ascendente es la misma que en la región descendente. La razón de crecimiento depende de $F$ y $\ell/k_d$. Para una $F$ dada, la razón de crecimiento será mayor cuanto mayor sea el parámetro de calentamiento $\varepsilon$.

## ABSTRACT

An analytical theory of two-level, moist, quasi-geostrophic baroclinic waves with meridional variation, but without the $\beta$-effect, is developed. The formulation is similar to that of Tang and Fichtl (1983) except that the meridional variation of the waves is allowed. The basic parameters are a rotational Froude number $F = 2f^2[S_d p_2^2(k_d^2+\ell^2)]^{-1}$ (where $f$ is the Coriolis parameter, $S_d$ the static stability in descending portion of the wave, $p_2$ the pressure at the middle level, $k_d$ the zonal wave number in the descending portion of the wave, $\ell(=\pi/D)$ the meridional wave number and $D$ the meridional extent of the wave) and a heating parameter $\varepsilon$ which is proportional to the midlevel vertical gradient of the mean flow saturation mixing ratio. For $\varepsilon \neq 0$ the disturbances are characterized by an unequal zonal length of the ascending or wet portion of the wave ($a$) and zonal length of the descending or dry portion of the wave ($b$). The first mode has a small region of strong ascending motion and a large region of weak descending motion ($a/b < 1$) with the reverse for the second mode ($a/b > 1$). These features are similar to those obtained by Tang and Fichtl (1983). In the present paper a meridional-scale equation is derived, expressing three possibilities: (i) $\ell/k_d = 0$ (no meridional variation of baroclinic waves, discussed in Tang and Fichtl, 1983); (ii) $\varepsilon = 0$ (dry model, discussed in Phillips, 1954, with $\beta$-effect ignored, $\ell$ being arbitrary); and (iii) a biquadratic equation in $\ell/k_d$. This latter equation essentially contains the information of the influence of latent heat release on the meridional scale of baroclinic waves. The model of meridionally uniform baroclinic waves ($\ell/k_d = 0$) and the dry model ($\varepsilon = 0$) are singular in that the characteristics of these two models cannot be deduced by setting $\ell/k_d \to 0$ or $\varepsilon \to 0$ in this biquadratic equation. For $\varepsilon < 0.464$, the ratio of the meridional extent $D$ of the zonal domain, $a + b$, is less than 0.7. For a given $\varepsilon$ and a given $F$, this ratio is larger for the first mode than for the second mode. The growth rate in the ascending region is equal to that in the descending region. The growth rate depends on both $F$ and $\ell/k_d$. For a given $F$, the larger the heating parameter $\varepsilon$, the larger the growth rate.

## 1. Introduction

A simple analytical quasi-geostrophic baroclinic model with latent heat release was investigated by Tang and Fichtl (hereafter referred to as TF), 1983. The second-order non-quasi-geostrophic effects were incorporated in TF (1984). An important result was that the width of the moist region can be different from that of the dry region. In fact, one of the two modes (i.e., the first mode) has a narrow moist region and a wide dry region. The growth rate in the moist region is equal to that in the dry region, but the growth in the moist region is different from that in the dry region (*cf.* TF, 1983). The non-quasi-geostrophic effects in the model with latent heat release reveal some features obtained in Saltzman and Tang (1972, 1975), e.g., in the first mode there is an intensification and contraction of the trough and the weakening and spreading of the ridge compared to the quasi-geostrophic case. It was also shown that the frontal zones were intensified by latent heat release in the non-quasi-geostrophic model. Another feature due to latent heat release in the ascending region (but no latent heating or evaporative cooling in the descending region) is the increase of the zonal mean temperature in the model atmosphere. Thus, TF (1983, 1984) reproduced some salient features of the observed zonal variation in the real atmosphere. But since the waves have no meridional variation, the influence of latent heat release on the meridional scale of baroclinic waves was excluded.

Now the meridional variation of baroclinic waves with latent heat release is allowed within the framework of a two-level, quasi-geostrophic model (Phillips, 1954), without the $\beta$-effect, for an atmosphere saturated with water vapor and subject to pseudo-adiabatic lifting and dry adiabatic subsidence. We shall see that the meridional scale of a class of baroclinic waves can be determined by latent heat release.

## 2. The governing equations

The set of the equations without $\beta$-effect are given by equation (6) of TF (1983). The set is

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + u_2\frac{\partial}{\partial x}\nabla^2\psi_2 + u_T\frac{\partial}{\partial x}\nabla^2\psi_T + v_2\frac{\partial}{\partial y}\nabla^2\psi_2 + v_T\frac{\partial}{\partial y}\nabla^2\psi_T = 0$$

$$\frac{\partial}{\partial t}\nabla^2\psi_T + u_2\frac{\partial}{\partial x}\nabla^2\psi_T + u_T\frac{\partial}{\partial x}\nabla^2\psi_2 + v_2\frac{\partial}{\partial y}\nabla^2\psi_T + v_T\frac{\partial}{\partial y}\nabla^2\psi_2 = \frac{f}{p_2}\omega_2$$

$$\frac{\partial}{\partial t}\psi_T + u_2\frac{\partial}{\partial x}\psi_T + v_2\frac{\partial}{\partial y}\psi_T - \frac{p_2 S}{2f}\omega_2 = 0 \tag{1}$$

where the symbols have usual meaning. Note that the subscripts 2 and T relate to the subscript 1 (upper level) and 3 (lower level) as follows:

$$()_2 = [()_1 + ()_3]/2 \tag{2}$$

$$()_T = [()_1 - ()_3]/2 \tag{3}$$

and $u = -\partial\psi/\partial y, v = \partial\psi/\partial y$. Note that $u$ and $v$ are geostrophic components of the wind.

$S$ is the static stability generalized to include the effect of latent heat release, i.e.:

$$S = S_d(1 - \delta\varepsilon) \tag{4}$$

where

$$\delta = \begin{cases} 1 & \text{for} \quad \omega_2 < 0 \\ 0 & \text{for} \quad \omega_2 > 0 \end{cases} \tag{5a}$$

$$S_d = -\frac{RT}{p}\partial \ell n\theta/\partial p \tag{5b}$$

$$\varepsilon = \frac{R}{S_d p_2}\frac{L}{c_p}(\frac{\partial q_s}{\partial p})_2 \tag{5c}$$

where both $S_d$ and $\varepsilon$ are constant, $T$ is temperature, $\theta$ potential temperature, $R$ gas constant for air (all for the dry atmosphere), $L$ the latent heat of evaporation of water, $c_p$ the specific heat at constant pressure, and $q_s$ the saturation mixing ratio.

Now we superimpose a disturbance (denoted by a prime) on a basic mean zonal flow. We shall use a frame of reference which translates with the mean zonal wind at the middle level. Thus, the coordinate transformation formulas can be written as

$$x_0 = x - \bar{u}_2 t$$

$$t_o = t \tag{6}$$

where $x_o$ and $t_o$ are the zonal and time coordinates in the moving frame of reference. The bar denotes a zonal average

$$\overline{(\ )} = \lim_{\Delta\to 0}\frac{1}{a+b}(\int_{-a+\Delta}^{0}(\tilde{\ })dx_0 + \int_0^{\Delta}(\hat{\ })dx_0 + \int_{\Delta}^{b}(\hat{\ })dx_0 + \int_b^{b+\Delta}(\tilde{\ })dx_o) \tag{7}$$

where the tilde indicates the quantity in the pseudo-adiabatically ascending region of zonal length $a$ and the caret indicates the dry-adiabatically descending region of zonal length $b$. Note that, in contrast to the classical dry model, disturbances characterized by $a \neq b$ are allowed. The increment $\Delta$ is introduced in (7) because the quantity represented by the parantheses may be discontinuous between the pseudo-adiabatically ascending region and dry adiabatically descending region.

Using (6),

$$\frac{\partial}{\partial t_0} = \frac{\partial}{\partial t} + \bar{u}_2\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x_0} = \frac{\partial}{\partial x} \tag{8}$$

The zonal mean equations can be obtained by taking the zonal average of (1) and using (8). They are

$$\frac{\partial}{\partial t_o}\frac{\partial \bar{u}_2}{\partial y} = \overline{\frac{\partial \psi_2'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2 \psi_2'}{\partial x_0^2} - \frac{\partial \psi_2'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2 \psi_2'}{\partial x_0^2} + \frac{\partial \psi_T'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2 \psi_T'}{\partial x_0^2} - \frac{\partial \psi_T'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2 \psi_T'}{\partial x_0^2}}$$

$$\frac{\partial}{\partial t_0}\frac{\partial \bar{u}_T}{\partial y} = \overline{\frac{\partial \psi_2'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2 \psi_T'}{\partial x_0^2} - \frac{\partial \psi_2'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2 \psi_T'}{\partial x_0^2} + \frac{\partial \psi_T'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2 \psi_2'}{\partial x_0^2} - \frac{\partial \psi_T'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2 \psi_2'}{\partial x_0^2}} - \frac{f}{p_2}\bar{\omega}_2 \tag{9}$$

$$\frac{\partial}{\partial t_0}\bar{\psi}_T = -\overline{\left(\frac{\partial \psi_2'}{\partial x_0}\frac{\partial \psi_T'}{\partial y} - \frac{\partial \psi_2'}{\partial y}\frac{\partial \psi_T'}{\partial x_0}\right)} + \frac{p_2 S_d \varepsilon}{2f(a+b)}\int_0^b \hat{\omega}_2' dx_0 + \frac{p_2}{2f}\bar{S}\bar{\omega}_2$$

where

$$\bar{S} = S_d\left(1 - \frac{a\varepsilon}{a+b}\right).$$

The eddy equations, with (8) applied, are as follows:

Dry region:

$$\frac{\partial}{\partial t_0}\left(\frac{\partial^2 \hat{\psi}_2'}{\partial x_0^2} - \ell^2 \hat{\psi}_2'\right) + \bar{u}_T\frac{\partial}{\partial x_0}\left(\frac{\partial^2 \hat{\psi}_T'}{\partial x_0^2} - \ell^2 \hat{\psi}_T'\right) - \frac{\partial^2 \bar{u}_2}{\partial y^2}\frac{\partial \hat{\psi}_2'}{\partial x_0} - \frac{\partial^2 \bar{u}_T}{\partial y^2}\frac{\partial \hat{\psi}_T'}{\partial x_0} = \hat{N}_2$$

$$\frac{\partial}{\partial t_0}\left(\frac{\partial^2 \hat{\psi}_T'}{\partial x_0^2} - \ell^2 \hat{\psi}_T'\right) + \bar{u}_T\frac{\partial}{\partial x_0}\left(\frac{\partial^2 \hat{\psi}_2'}{\partial x_0^2} - \ell^2 \hat{\psi}_2'\right) - \frac{\partial^2 \bar{u}_T}{\partial y^2}\frac{\partial \hat{\psi}_2'}{\partial x_0} - \frac{\partial^2 \bar{u}_2}{\partial y^2}\frac{\partial \hat{\psi}_T'}{\partial x_0} = \frac{f}{p_2}\hat{\omega}_2' + \hat{N}_T \tag{10}$$

$$\frac{\partial \hat{\psi}_T'}{\partial t_0} - \bar{u}_T\frac{\partial \hat{\psi}_T'}{\partial x_0} - \frac{p_2 S_d}{2f}\left[\hat{\omega}_2' - \frac{\varepsilon}{(a+b)}\int_0^b \hat{\omega}_2' dx_0\right] = \hat{M}_T$$

Moist region:

$$\frac{\partial}{\partial t_0}\left(\frac{\partial^2 \tilde{\psi}_2'}{\partial x_0^2} - \ell^2 \tilde{\psi}_2'\right) + \bar{u}_T\frac{\partial}{\partial x_0}\left(\frac{\partial^2 \tilde{\psi}_T'}{\partial x_0^2} - \ell^2 \tilde{\psi}_T'\right) - \frac{\partial^2 \bar{u}_2}{\partial y^2}\frac{\partial \tilde{\psi}_2'}{\partial x_0} - \frac{\partial^2 \bar{u}_T}{\partial y^2}\frac{\partial \tilde{\psi}_T'}{\partial x_0} = \tilde{N}_2$$

$$\frac{\partial}{\partial t_0}\left(\frac{\partial^2 \tilde{\psi}_T'}{\partial x_0^2} - \ell^2 \tilde{\psi}_T'\right) + \bar{u}_T\frac{\partial}{\partial x_0}\left(\frac{\partial^2 \tilde{\psi}_2'}{\partial x_0^2} - \ell^2 \tilde{\psi}_2'\right) - \frac{\partial^2 \bar{u}_T}{\partial y^2}\frac{\partial \tilde{\psi}_2'}{\partial x_0} - \frac{\partial^2 \bar{u}_2}{\partial y^2}\frac{\partial \tilde{\psi}_T'}{\partial x_0} = \frac{f}{p_2}\tilde{\omega}_2' + \tilde{N}_T \tag{11}$$

$$\frac{\partial \tilde{\psi}_T'}{\partial t_0} - \bar{u}_T\frac{\partial \tilde{\psi}_T'}{\partial x_0} - \frac{p_2 S_d}{2f}\left[(1-\varepsilon)\tilde{\omega}_2' + \frac{\varepsilon}{(a+b)}\int_{-a}^0 \tilde{\omega}_2' dx_0\right] = \tilde{M}_T$$

where

$$N_2 = -(\frac{\partial\psi_2'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2\psi_2'}{\partial x_0^2} + \frac{\partial\psi_T'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2\psi_T'}{\partial x_0^2} - \frac{\partial\psi_2'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2\psi_2'}{\partial x_0^2} - \frac{\partial\psi_T'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2\psi_T'}{\partial x_0^2})'$$

$$N_T = -(\frac{\partial\psi_2'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2\psi_T'}{\partial x_0^2} + \frac{\partial\psi_T'}{\partial x_0}\frac{\partial}{\partial y}\frac{\partial^2\psi_2'}{\partial x_0^2} - \frac{\partial\psi_2'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2\psi_T'}{\partial x_0^2} - \frac{\partial\psi_T'}{\partial y}\frac{\partial}{\partial x_0}\frac{\partial^2\psi_2'}{\partial x_0^2})' \tag{12}$$

$$M_T = -(\frac{\partial\psi_2'}{\partial x_0}\frac{\partial\psi_T'}{\partial y} - \frac{\partial\psi_2'}{\partial y}\frac{\partial\psi_T'}{\partial x_0})' .$$

In addition, we have the mass balance requirement

$$\int_0^D\int_{-a}^0 \tilde{\omega}_2' dx_0 + \int_0^D\int_0^b \hat{\omega}_2' dx_0 = 0. \tag{13}$$

## 3. Linearized baroclinic system with static stability/vertical motion correlation

Now we consider the baroclinic case, i. e.:

$$\frac{\partial^2\bar{u}_2}{\partial y^2} = \frac{\partial^2\bar{u}_T}{\partial y^2} = 0 \tag{14}$$

and we assume that the perturbation is small compared to the mean field. Thus, we set

$$N_2 = N_T = M_T = 0. \tag{15}$$

This means that the nonlinear quantities in the advection terms are ignored. Similar to Phillips (1954) we assumed that the meridional variation of the perturbation is of the form *sin* $\ell y$, i. e.:

$$\psi_2', \psi_T', \omega_2' \sim sin\ \ell y \tag{16a}$$

$$\ell = \frac{\pi}{D} \tag{16b}$$

with (14), (15) and (16) substituted (11), (12) and (13) become:

Dry region:

$$\frac{\partial}{\partial t_o}(\frac{\partial^2\hat{\psi}_2'}{\partial x_0^2} - \ell^2\hat{\psi}_2') + \bar{u}_T\frac{\partial}{\partial x_0}(\frac{\partial^2\hat{\psi}_T'}{\partial x_0^2} - \ell^2\hat{\psi}_T') = 0$$

$$\frac{\partial}{\partial t_0}(\frac{\partial^2\hat{\psi}_T'}{\partial x_0^2} - \ell^2\hat{\psi}_T') + \bar{u}_T\frac{\partial}{\partial x_0}(\frac{\partial^2\hat{\psi}_2'}{\partial x_0^2} - \ell^2\hat{\psi}_2') = \frac{f}{p_2}\hat{\omega}_2' \tag{17}$$

$$\frac{\partial\hat{\psi}_T'}{\partial t_0} - \bar{u}_T\frac{\partial\hat{\psi}_T'}{\partial x_0} - \frac{p_2 S_d}{2f}[\hat{\omega}_2' - \frac{\varepsilon}{(a+b)}\int_0^b \hat{\omega}_2' dx_0] = 0$$

Moist region:

$$\frac{\partial}{\partial t_0}\left(\frac{\partial^2 \tilde{\psi}_2'}{\partial x_0^2} - \ell^2 \tilde{\psi}_2'\right) + \bar{u}_T \frac{\partial}{\partial x_0}\left(\frac{\partial^2 \tilde{\psi}_T'}{\partial x_0^2} - \ell^2 \tilde{\psi}_T'\right) = 0$$

$$\frac{\partial}{\partial t_0}\left(\frac{\partial^2 \tilde{\psi}_T'}{\partial x_0^2} - \ell^2 \tilde{\psi}_T'\right) + \bar{u}_T \frac{\partial}{\partial x_0}\left(\frac{\partial^2 \tilde{\psi}_2'}{\partial x_0^2} - \ell^2 \tilde{\psi}_2'\right) = \frac{f}{p_2}\tilde{\omega}_2' \tag{18}$$

$$\frac{\partial \tilde{\psi}_T'}{\partial t_0} - \bar{u}_T \frac{\partial \tilde{\psi}_T'}{\partial x_0} - \frac{p_2 S_d}{2f}\left[(1-\epsilon)\tilde{\omega}_2' + \frac{\epsilon}{(a+b)}\int_{-a}^{0} \tilde{\omega}_2' dx_0\right] = 0\ .$$

Mass balance requirement:

$$\int_{-a}^{0} \tilde{\omega}_2' dx_0 + \int_{0}^{b} \hat{\omega}_2' dx_0 = 0\ . \tag{19}$$

The basic wind shear in (17) and (18) is constant in space and time. The system of equations is linear in each region, because in additon to the fact that the nonlinear part of the advection terms are ignored, the static stability fluctuations are constat in each region. The combined system of equations (17) and (18), constitutes a nonlinear set because the form of the thermodynamic equation depends on the sign of $\omega$ (*cf.* TF, 1983).

## 4. Stability analysis and meridional scale of the moist baroclinic waves

Assume that the disturbance grows exponentially with time in the frame of reference moving with the mean zonal wind at the middle level, such that

$$\frac{\partial}{\partial t_0}()' = \nu()' \tag{20}$$

where $\nu$ is the gowth rate. Because of the mass balance requirement (19), the growth rate in the moist region is equal to that in the dry region. Similar to the method used in TF (1983) we can derive a single equation for $\hat{\omega}_2'$ in the dry region and for $\tilde{\omega}_2'$ in the moist region. They are as follows:

Dry region:

$$\hat{\omega}'_{2x_0x_0x_0x_0} + (\lambda_d - \ell^2 - n^2)\hat{\omega}'_{2x_0x_0} + n^2(\lambda_d + \ell^2)\hat{\omega}_2' - n^2\ell^2 \frac{\epsilon}{(a+b)}\int_0^b \hat{\omega}_2' dx_0 = 0 \tag{21}$$

Moist region:

$$\tilde{\omega}'_{2x_0x_0x_0x_0} + (\lambda_m - \ell^2 - n^2)\tilde{\omega}'_{2x_0x_0} + n^2(\lambda_m + \ell^2)\tilde{\omega}_2' + n^2\ell^2 \frac{\epsilon}{(1-\epsilon)(a+b)}\int_{-a}^{0} \tilde{\omega}_2' dx_0 = 0 \tag{22}$$

where the subscript $x_0$ indicates differentiation with respect to $x_0$ and other parameters are given by

$$\lambda_d = 2f^2/S_d p_2^2 \tag{23a}$$

$$\lambda_m = \lambda_d/(1-\varepsilon) \tag{23b}$$

$$n = \nu/\bar{u}_T. \tag{23c}$$

The solution of the differential equations (21) and (22) for the vertical motion can be written as (see Appendix)

$$\hat{\omega}_2' = (A_d sin k_d x_0 + B_d cos k_d x_0 + C_d)e^{\nu t_0} sin\ \ell y \tag{24a}$$

$$\tilde{\omega}_2' = (A_m sin k_m x_0 + B_m cos k_m x_0 + C_m)e^{\nu t_0} sin\ \ell y. \tag{24b}$$

Substitution of (24$a$) in (21) and (24$b$) in (22) leads to

$$(\frac{n}{k_d})^2 = \frac{F-1}{F+1} = (\frac{k_m}{k_d})^2(\frac{\gamma F - K}{\gamma F + K}) \tag{25}$$

$$C_d = \hat{\alpha}[A_d(1 - cos k_d b) + B_d sin k_d b] \tag{26}$$

$$C_m = \tilde{\alpha}[A_m(1 - cos k_m a) - B_m sin k_m a] \tag{27}$$

where

$$F = \lambda_d/(k_d^2 + \ell^2) \tag{28a}$$

$$\gamma = \lambda_m/\lambda_d = (1-\varepsilon)^{-1} \tag{28b}$$

$$K = (k_m^2 + \ell^2)/(k_d^2 + \ell^2) \tag{28c}$$

$$\hat{\alpha} = \frac{\varepsilon}{k_d b\{[1 + \frac{a}{b}][(\frac{k_d}{\ell})^2 F + F + 1] - \varepsilon\}} \tag{28d}$$

$$\tilde{\alpha} = \hat{\alpha}/(k_m/k_d). \tag{28e}$$

From the definition of K in (28c) we obtain

$$k_m/k_d = [K + (K - 1)(\ell/k_d)^2]^{1/2} \tag{29}$$

substitution of (29) in (25) with the aid of (23a), (23c) and (28a) leads to the formula for the growth rate, i.e.:

$$\frac{\nu}{\sqrt{\lambda_d}\bar{u}_T} = \{\frac{1}{F}[\frac{F-1}{F+1}]/[1+(\frac{\ell}{k_d})^2]\}^{\frac{1}{2}}$$

$$= \{[K+(K-1)(\frac{\ell}{k_d})^2]\frac{1}{F}[\frac{\gamma F-K}{\gamma F+K}]/[1+(\frac{\ell}{k_d})^2]\}^{\frac{1}{2}}. \qquad (30)$$

From the second expression of (30) we obtain a quadratic equation in $K$, i.e.:

$$K^2[1+(\frac{\ell}{k_d})^2]+K[\frac{F-1}{F+1}-\gamma F-(\frac{\ell}{k_d})^2(\gamma F+1)]+\gamma F[\frac{F-1}{F+1}+(\frac{\ell}{k_d})^2]=0 \qquad (31)$$

with the roots given by

$$K=\{\frac{1}{2}[\gamma F+(\frac{\ell}{k_d})^2(\gamma F+1)-\frac{F-1}{F+1}]$$

$$\pm[\frac{1}{4}(\gamma F+(\frac{\ell}{k_d})^2(\gamma F+1)-\frac{F-1}{F+1})^2-\gamma F(\frac{F-1}{F+1}+(\frac{\ell}{k_d})^2)(1+(\frac{\ell}{k_d})^2)]^{\frac{1}{2}}\}\{1+(\frac{\ell}{k_d})^2\}^{-1}. \qquad (32)$$

Recall from (24a, b, 26 and 27) that $A_d, B_d, A_m$ and $B_m$ are not determined. The boundary conditions for $\omega_2'$

$$\hat{\omega}_2'=0 \text{ at } x_0=0,b$$

$$\tilde{\omega}_2'=0 \text{ at } x_0=-a,0 \qquad (33)$$

are needed to find the ratios, $B_d/A_d, B_m/A_m$ and other parameter relations. When (24a, b) are substituted in (33) with the aid of (26) and (27), we obtain

$$\frac{B_d}{A_d}=-\frac{\hat{\alpha}(1-\cos k_d b)}{1+\hat{\alpha}\sin k_d b} \qquad (34a)$$

$$\frac{B_m}{A_m}=-\frac{\tilde{\alpha}(1-\cos k_m a)}{1-\tilde{\alpha}\sin k_m a} \qquad (34b)$$

$$\tan(k_d b/2)=-(2\hat{\alpha})^{-1} \qquad (34c)$$

$$\tan(k_m a/2)=-(2\tilde{\alpha})^{-1} \qquad (34d)$$

Substitution of (34a, b) in (24a, b), (26) and (27) and substitution of (26) and (27) in (24a, b) lead to

$$\hat{\omega}_2' = A_d[sink_d x_0 + (1 - cosk_d x_0)\frac{\hat{\alpha}(1 - cosk_d b)}{1 + \hat{\alpha} sink_d b}]e^{\nu t_0} sin\ \ell y \tag{35a}$$

$$\tilde{\omega}_2' = A_m[sink_m x_0 + (1 - cosk_m x_0)\frac{\tilde{\alpha}(1 - cosk_m a)}{1 - \tilde{\alpha} sink_m a}]e^{\nu t_0} sin\ \ell y. \tag{35b}$$

Substitution of (35a, b) in the mass balance requirement yields the formula for the ratio $A_m/A_d$, i. e.:

$$\frac{A_m}{A_d} = (\frac{k_m}{k_d})(\frac{1 + \hat{\alpha} k_d b}{1 - \tilde{\alpha} k_m a})(\frac{1 - \tilde{\alpha} sink_m a}{1 + \hat{\alpha} sink_d b})(\frac{1 - cosk_d b}{1 - cosk_m a}). \tag{36}$$

From (35a, b) we can obtain the divergent components of the wind by using the continuity equations. They are

$$\hat{u}_\chi' = \frac{k_d A_d}{p_2(k_d^2 + \ell^2)}[cosk_d x_0 + \frac{\hat{\alpha}(1 - cosk_d b)}{1 + \hat{\alpha} sink_d b} sink_d x_0]e^{\nu t_0} sin\ \ell y \tag{37a}$$

$$\tilde{u}_\chi' = \frac{k_m A_m}{p_2(k_m^2 + \ell^2)}[cosk_m x_0 + \frac{\tilde{\alpha}(1 - cosk_m a)}{1 - \tilde{\alpha} sink_m a} sink_m x_0]e^{\nu t_0} sin\ \ell y \tag{37b}$$

$$\hat{v}_\chi' = \frac{\ell A_d}{p_2(k_d^2 + \ell^2)}[sink_d x_0 + (\frac{k_d^2 + \ell^2}{\ell^2} - cosk_d x_0)\frac{\hat{\alpha}(1 - cosk_d b)}{1 + \hat{\alpha} sink_d b}]e^{\nu t_0} cos\ \ell y \tag{38a}$$

$$\tilde{v}_\chi' = \frac{\ell A_m}{p_2(k_m^2 + \ell^2)}[sink_m x_0 + (\frac{k_m^2 + \ell^2}{\ell^2} - cosk_m x_0)\frac{\tilde{\alpha}(1 - cosk_m a)}{1 - \tilde{\alpha} sink_m a}]e^{\nu t_0} cos\ \ell y \tag{38b}$$

where

$$u_\chi' = u_{\chi 1}' = -u_{\chi 3}'$$

$$v_\chi' = v_{\chi 1}' = -v_{\chi 3}' \tag{39}$$

Next, we apply the condition that the normal component of the wind across the meridional interface between the pseudo-adiabatically ascending region and dry adiabatically descending region must be continuous, and that this condition applies to the periodic meridional interface. Since $\overline{u}$ is a constant at any given level, $\overline{u}$ is continuous across meridional interface. Thus, $u' + u_\chi'$ must be continuous at such an interface. However, the non-divergent component $u'$ vanishes at the reference latitude $y = D/2$. Therefore, we have

$$\hat{u}_\chi' = \tilde{u}_\chi' \text{ at } x_0 = 0, y = D/2 \tag{40a}$$

$$\hat{u}_\chi'(x_0 = b) = \tilde{u}_\chi'(x_0 = -a), y = D/2. \tag{40b}$$

Note, that, because of the assumed form of meridional variation in (37*a*, *b*), the conditions in (40*a*, *b*) also apply at an arbitrary latitude. Applying (37*a*,*b*) in (40*a*, *b*), we obtain

$$A_m/A_d = Kk_d/k_m \tag{41}$$

$$(\frac{1-\cos k_d b}{1-\cos k_m a})(\frac{1-\tilde{\alpha}\sin k_m a}{1+\hat{\alpha}\sin k_d b}) = 1. \tag{42}$$

Substitution of (42) in (36) leads to

$$\frac{A_m}{A_d} = (\frac{k_m}{k_d})(\frac{1+\hat{\alpha}k_d b}{1-\tilde{\alpha}k_m a}). \tag{43}$$

From (41) and (43) with the aid of (28d, e) and (29) we obtain

$$(\frac{\ell}{k_d})^2\{K[\gamma(F+1+(\frac{\ell}{k_d})^2)-1]-\gamma[F+(\frac{\ell}{k_d})^2]\}=0. \tag{44}$$

Elimination of $K$ between (44) and (31) gives

$$(\frac{\ell}{k_d})^2\frac{\varepsilon}{(1-\varepsilon)^2}\{[\gamma F^2-(1+\gamma)F-1](\frac{\ell}{k_d})^4+[\gamma F^3-(2+\gamma)F^2-(1+2\gamma)F-1](\frac{\ell}{k_d})^2-\gamma F(F+1)^2\}=0 \tag{45}$$

From (45) there are three possibilities:

(i) $(\ell/k_d)^2 = 0$. This is the case that the perturbation has no meridional variation, which was investigated in TF (1983).

(ii) $\varepsilon = 0$. This is the dry model which has been studied many times.

(iii) The quantity in the large brackets vanishes. This is the new result. It is a bi-quadratic equation in $\ell/k_d$, whose solution can be written as

$$\frac{\ell}{k_d} = \{-\frac{a_2}{2a_1}+[(\frac{a_2}{2a_1})^2-a_3]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{46}$$

where

$$a_1 = \gamma F^2-(1+\gamma)F-1$$

$$a_2 = \gamma F^3-(2+\gamma)F^2-(1+2\gamma)F-1 \tag{47}$$

$$a_3 = -\gamma F(F+1)^2$$

The minus sign in front of the brackets in (46) is deleted because it gives an imaginary value of $\ell/k_d$. Note that from (ii) the meridional wave number cannot be determined: since the meridional wave number is inversely proportional to the meridional extent of the domain, the meridional scale of the disturbance cannot be determined in the dry model. On the other hand, from (46) we see that the meridional scale of the moist baroclinic waves can be determined by latent heat release.

Next, we wish to derive a formula for $a/b$. Substitution of (28d) in (34c) gives

$$\frac{a}{b} = -\{F + (F+1-\varepsilon)(\frac{\ell}{k_d})^2 + \varepsilon(\frac{\ell}{k_d})^2[(tan\frac{k_d b}{2})/\frac{k_d b}{2}]\}/\{F + (F+1)(\frac{\ell}{k_d})^2\}. \quad (48)$$

Similarly, substitution of (28e) in (34d) with the aid of (28d) yields

$$\frac{b}{a} = \{\varepsilon(\frac{\ell}{k_d})^2[(tan\frac{k_m a}{2})/\frac{k_m a}{2}] - [F + (F+1)(\frac{\ell}{k_d})^2]\}/\{F + (F+1-\varepsilon)(\frac{\ell}{k_d})^2\}. \quad (49)$$

But, $k_d b$ in (48) and $k_m a$ in (49) are unknown. From (28e), (34c) and (34d) we get

$$tan\frac{k_m a}{2} = -(\frac{k_m}{k_d})tan\frac{k_d b}{2}. \quad (50)$$

The product of (48) and (49) together with the expression given by (50) yields

$$(\frac{k_m}{k_d})\{\varepsilon(\frac{\ell}{k_d})^2 tan\frac{k_d b}{2} + (\frac{k_d b}{2})[F + (F+1-\varepsilon)(\frac{\ell}{k_d})^2]\}$$

$$-[F + (F+1-\varepsilon)(\frac{\ell}{k_d})^2]tan^{-1}[(\frac{k_m}{k_d})tan\frac{k_d b}{2}] = 0 \quad (51)$$

where the formula for the $k_m/k_d$ is given by (29). From (51) using (46) and (29) we can obtain the value of $k_d b$ by iteration. From the values of $k_d b$ and $\ell/k_d$ we can use (48) to obtain the ratio of zonal length of moist region to that of dry region, $a/b$. From the values of $k_d b, \ell/k_d$ and $a/b$, the ratio of the meridional extent to the zonal domain can be computed using the formula

$$\frac{D}{a+b} = \pi/\{k_d b[1 + (a/b)][\ell/k_d]\}. \quad (52)$$

We are now in a position to calculate the growth rate, the ratio of the area of the moist region to that of the dry region, the ratio of the meridional extent to the zonal domain. The figures shown in the following are for $\varepsilon = 0.2$, 0.3 and 0.464. From (46) we calculate $\ell/k_d$. Figure 1 shows the curves for $\ell/k_d$ as functions of the rotational Froude number. As $\varepsilon$ decreases, $\ell/k_d$ increases, but the range of $F$ in which the solution is valid becomes narrower. From the first expression of (30) the growth rate depends on both $F$ and $\ell/k_d$. But $\ell/k_d$ depends on the latent heating parameter $\varepsilon$ as seen in Figure 1. Thus, the growth rate is a function of both $F$ and $\varepsilon$. Figure 2 shows the growth rate (measured in the unit of $\lambda_d^{1/2}\overline{u}_T$) versus $F$. For a given $F$, the larger the latent heat release (i. e., the larger $\varepsilon$), the larger the growth rate. Next, we calculate $K$ from (32) and then calculate $k_m/k_d$ from (29). Figure 3 shows the values of $k_m/k_d$ versus $F$; the upper curves correspond to plus sign in front of the square root in the formula for $K$ in (32), and the lower curves correspond to minus sign. We shall call the former the first mode and the latter the second mode. Next, we calculate $a/b$ from (48). In this formulation the meridional extent of the moist region is equal to that of the dry region. Thus $a/b$ is also equal to the ratio of moist area to dry area. This is shown in Figure 4. Similar to TF (1983), the first mode has a narrow ascending region and wide descending region, and the second mode has a wide ascending region and narrow descending region. But the first mode in this model has narrower ascending region than that in TF (1983) for given $\varepsilon$ and $F$. Next, the ratio of the meridional extent to the zonal domain is calculated using (52) and is shown in Figure 5. For given $\varepsilon$ and $F$, this ratio is larger for the first mode than for the second mode. For $\varepsilon \leq 0.464$, this ratio is less than 0.7. For a given $F$ in the same mode, the larger the heating parameter $\varepsilon$ the larger the ratio.

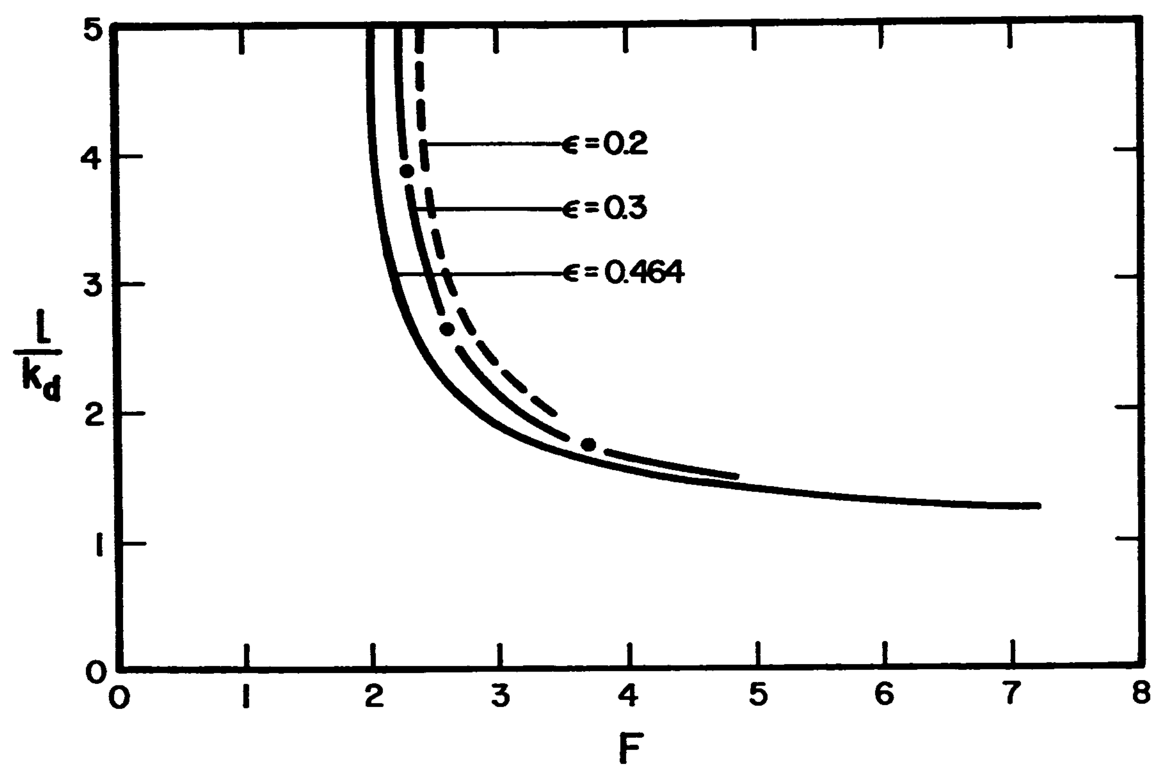


Fig. 1. The ratio of the meridional wave number to the zonal wave number in the dry region, $\ell/k_d$, versus the rotational Froude number, $F$, for the heating parameter $\varepsilon = 0.2, 0.3$ and $0.464$.

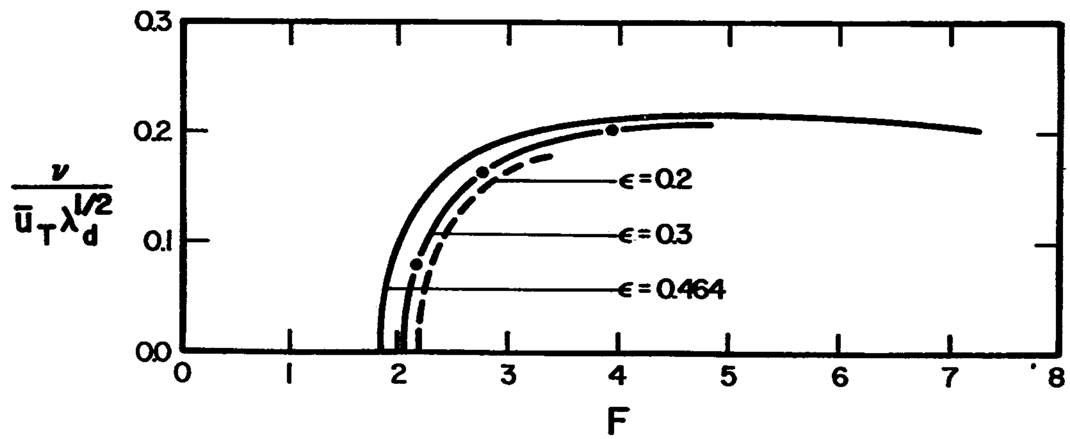


Fig. 2. The non-dimensional growth rate $\nu/\lambda_d^{1/2}\bar{u}_T$ versus $F$ for $\epsilon = 0.2, 0.3$ and $0.464$.

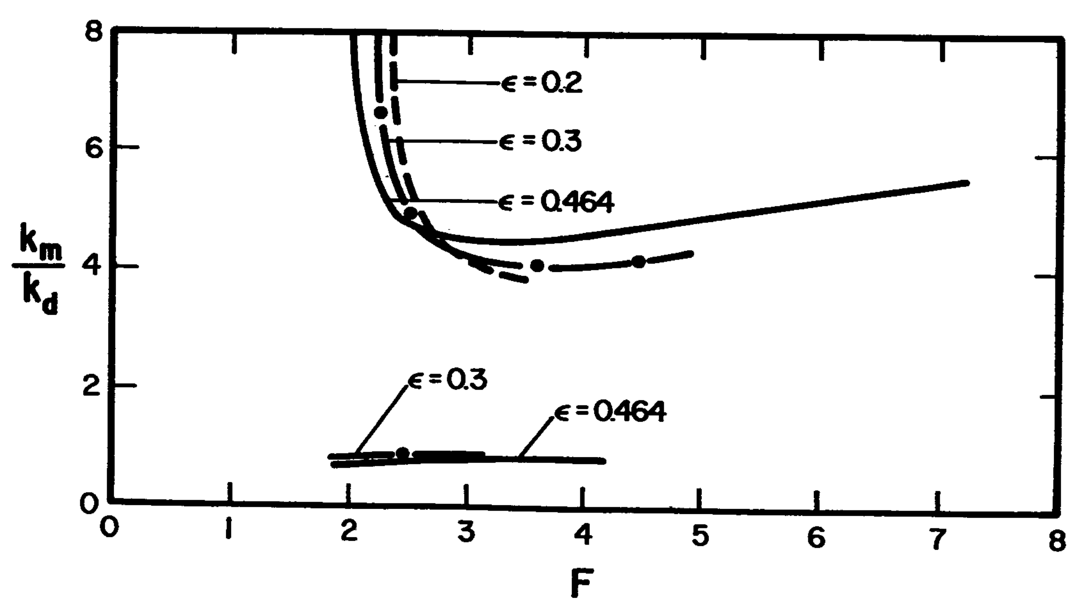


Fig. 3. $k_m/k_d$ vs. $F$ for $\epsilon = 0.2, 0.3$ and $0.464$. The upper set of curves represents the first mode and the lower set of curves represents the second mode.

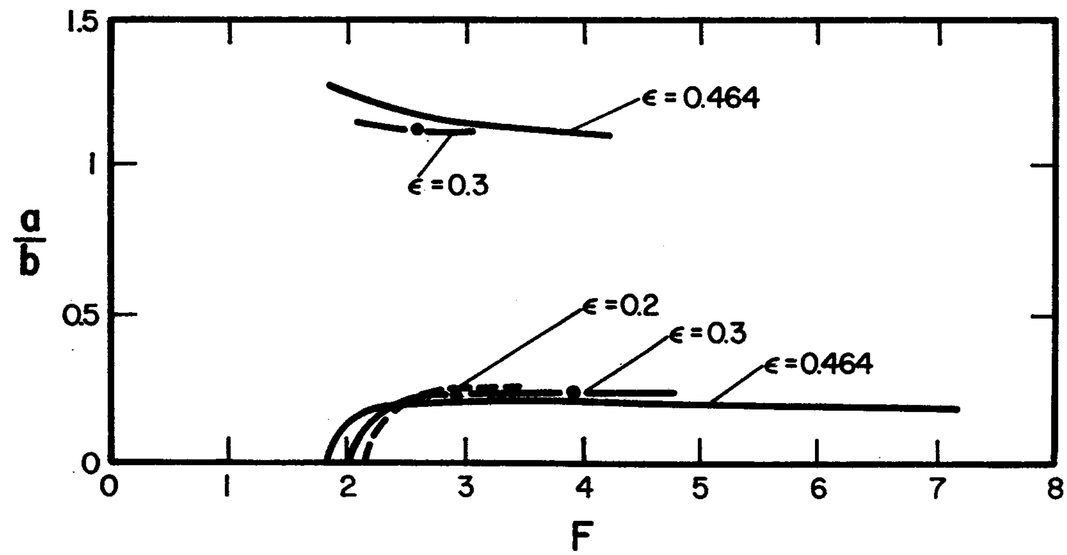


**Fig. 4.** The ratio of the zonal length of the moist region to that of the dry region, $a/b$, versus $F$ for $\epsilon = 0.2, 0.3$ and $0.464$. The lower set of curves represents the first mode and the upper set of curves represents the second mode.

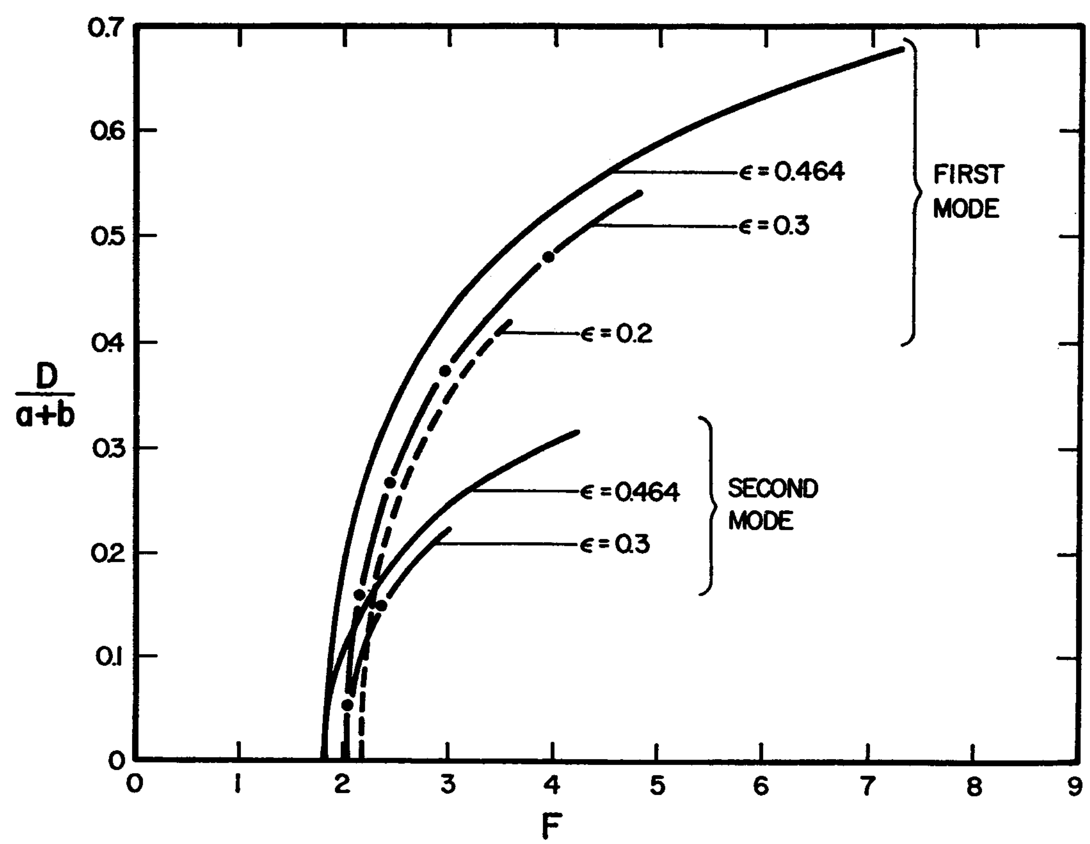


**Fig. 5.** The ratio of the meridional extent to the zonal domain, $D/(a+b)$, versus $F$ for $\epsilon = 0.2, 0.3$ and $0.464$.

## 5. Conclusions

We have shown the control on the meridional scale of a class of the baroclinic waves exercised by latent heat release. A meridional-scale equation was derived, i. e. (45), in which the dry model and the moist model without the meridional variation of the baroclinic waves were revealed. In the dry model, the stability analysis cannot determine the meridional scale of the baroclinic waves. When latent heat release is included, the meridional variation of the waves either vanishes (TF, 1983) or is finite. When the waves have meridional variation with latent heat release, the growth rate increases as the heating increases for a given rotational Froude number, and there are two modes: the first mode has small ascending region and large descending region while the second mode has small descending region and large ascending region. For the rotational Froude number greater than 2.5, the first mode shows that the zonal length in the moist region is about 20% of that in the dry region. This seems to agree qualitatively with the observed narrow cloud band along cold front on the saltellite images.

## Acknowledgements

I am very grateful to Dr. George H. Fichtl, Chief of the Division of Earth Science and Applications, National Aeronautics and Space Administration/Marshall Space Flight Center, Al, for suggesting the possibility of the control exercised by latent heat release on the meridional scale of baroclinic waves and for suggesting the form of the meridional variation of the waves. I thank Professor Barry Saltzman of Yale University for discussion of this work. This research was supported by the National Aeronautics and Space Administration under contract NAS8-36356 at Yale University and NAS8-37130 at the Universities Space Research Association.

## Apendix

The general solution to the equations (21) and (22) can be written as:

$$\hat{\omega}_2' = (G_d sin k_d x_0 + H_d cos k_d x_0 + I_d sin m_d k_d x_0 + J_d cos m_d k_d x_0 + C_d) e^{\nu t_0} sin\ \ell y \tag{A1}$$

$$\tilde{\omega}_2' = (G_m sin k_m x_0 + H_m cos k_m x_0 + I_m sin m_m k_m x_0 + J_m cos m_m k_m x_0 + C_m) e^{\nu t_0} sin\ \ell y \tag{A2}$$

Substitution of (A1) in (21) and substitution of (A2) in (22) give two sets of expressions:

$$k_d^4 - k_d^2(\lambda_d - \ell^2 - n^2) + n^2(\lambda_d + \ell^2) = 0 \tag{A3a}$$

$$m_d^4 k_d^4 - m_d^2 k_d^2(\lambda_d - \ell^2 - n^2) + n^2(\lambda_d + \ell^2) = 0 \tag{A3b}$$

$$n^2(\lambda_d + \ell^2)C_d - \frac{n^2\ell^2\epsilon}{k_d(a+b)}[-G_d(cos k_d b - 1) + H_d sin k_d b$$

$$-\frac{1}{m_d} I_d(cos m_d k_d b - 1) + \frac{1}{m_d} J_d sin m_d k_d b + k_d b C_d] = 0 \tag{A3c}$$

$$k_m^4 - k_m^2(\lambda_m - \ell^2 - n^2) + n^2(\lambda_m + \ell^2) = 0 \tag{A4a}$$

$$m_m^4 k_m^4 - m_m^2 k_m^2(\lambda_m - \ell^2 - n^2) + n^2(\lambda_m + \ell^2) = 0 \tag{A4b}$$

$$n^2(\lambda_m + \ell^2)C_m + \frac{n^2\ell^2\varepsilon}{k_m(a+b)(1-\varepsilon)}[-G_m(1 - cosk_m a) + H_m sink_m a$$
$$-\frac{1}{m_m}I_m(1 - cosm_m k_m a) + \frac{1}{m_m}J_m sinm_m k_m a + k_m a C_m] = 0. \tag{A4c}$$

The difference of (A3b) and (A3a) gives

$$m_d = \pm 1 \tag{A5a}$$

or

$$m_d = \pm\frac{1}{k_d}[\lambda_d - (k_d^2 + \ell^2) - n^2]^{\frac{1}{2}}. \tag{A5b}$$

Similarly, the difference of (A4b) and (A4a) gives

$$m_m = \pm 1 \tag{A6a}$$

or

$$m_m = \pm\frac{1}{k_m}[\lambda_m - (k_m^2 + \ell^2) - n^2]^{\frac{1}{2}}. \tag{A6b}$$

In order to limit the scope of this paper, we shall consider only (A5a) and (A6a), i. e.:

$$m_d = m_m = \pm 1.$$

Then, we can substitute (A5a) in (A1) and substitute (A6a) in (A2), and define

$$A_d = G_d \pm I_d$$

$$B_d = H_d + J_d$$

$$A_m = G_m \pm I_m$$

$$B_m = H_m + J_m.$$

Thus, we obtain (24A) and (24B).

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