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Radiative decay of a localized temperature perturbation in an idealized atmosphere

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RESUMEN

Utilizamos análisis de Fourier y los resultados de Spiegel (1957) para calcular el decaimiento radiativo de una perturbación de temperatura, pequeña, lisa y localizada en una atmósfera homogénea e isoterma bajo la suposición gris. Encontramos expresiones aproximadas analíticas para el decaimiento, tanto para los límites ópticamente delgado como grueso. Asimismo derivamos un método para checar la exactitud de la solución o soluciones, utilizando la conservación de la energía de la perturbación.

ABSTRACT

We use Fourier analysis and Spiegel's results (1957) to calculate the radiative decay of a small, smooth, localized, temperature perturbation in a homogeneous, isothermal atmosphere, assuming the grey case. We find approximate, analytic expressions for the decay in both the optically thin and optically thick limits. We also derive a method for checking the accuracy of the solution(s) using the conservation of energy of the perturbation.

1. Introduction

The radiative decay of sinusoidal temperature perturbations has been extensively investigated, partly because such structures are so readily generated by various wave phenomena in planetary and solar atmospheres, and partly because they are generally simpler than localized perturbations to understand from a mathematical veiwpoint. Spiegel (1957) conducted the first rigorous derivation of the linear radiative decay of a sinusoidal temperature perturbation. He examined the restricted problem of an infinite, homogeneous, isothermal, grey atmosphere, and found the decay rate to be dependent on the scale of the perturbation. Sasamori and London (1966) extended this work to a non-grey, plane parallel atmosphere of semi-infinite extent, using several line and band profiles, even the simplest of which required numerical techniques.

Gay (1978), Nagirner (1979) and Gay and Thomas (1980, 1982) showed that the effects of a finite boundary are to impose discrete eigenfrequencies on the decay modes. Due to the presence of boundary conditions the eigenfunctions of the controlling integral equation are *not* pure sinusoids, in analogy to the eigenfunctions of Schrödinger's equation in quantum mechanics. They discussed how other complicating influences may be included in the calculation of radiative cooling by exploiting the completeness properties of the eigenfunctions.

The mathematical expressions for radiative relaxation in a realistic planetary atmosphere often obscure the conceptual aspects. In this paper we consider an initial value problem in which a grey, infinite, homogeneous atmosphere is subjected to a temperature "pulse" of finite extent. It may be considered to be an extension of the work of Spiegel to a more realistic case in which *all* spatial frequencies are present.

In section 2 we develop the necessary equations and obtain the closed form solution for the temperature profile at all points in space and time. In section 3 we evaluate the solution numerically, in both the optically thick- and optically-thin limits. In section 4 we show how the sum of the internal energy of the gas and the radiative energy may be evaluated. From conservation of energy, this quantity is invariant in time, and therefore may be used as a check on the numerical accuracy of the solution. In the last section we discuss the results and make recommendations for future studies.

2. Development

We begin with the radiative transfer equation

$$-\frac{dI_\nu}{ds} = k_\nu^m \rho_a (I_\nu - S_\nu) \tag{1}$$

I_ν = spectral intensity,

s = distance along the light path,

k_ν^m = mass extinction coefficient,

ρ_a = mass density of the absorbing gas,

S_ν = spectral source function.

With the usual convention for a plane parallel atmosphere, we write the intensity in terms of upward (I_ν^+) and downward (I_ν^-) components. Expressions for these two intensities can be found from the formal solution to the radiative transfer equation for a plane parallel atmosphere. In an infinite medium there is no contribution from the boundaries, so the upward and downward intensities are:

$$I_\nu^+(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} d\tau_\nu' \frac{B_\nu(\tau_\nu')}{\mu} e^{-(\tau_\nu' - \tau_\nu)/\mu}, \tag{2a}$$

$$I_\nu^-(\tau_\nu, \mu) = \int_{-\infty}^{\tau_\nu} d\tau_\nu' \frac{B_\nu(\tau_\nu')}{\mu} e^{-(\tau_\nu - \tau_\nu')/\mu}, \tag{2b}$$

$$\tau_\nu = \int_o^z k_\nu^m \rho_a dz \tag{3}$$

is the vertical optical depth referenced to a plane at $z = 0$, which is at an arbitrary location.

The equation governing a localized temperature perturbation can then be obtained starting with the definition of the spectral heating rate:

$$Q_\nu = -\nabla \cdot \mathbf{F}_\nu = -\int_{4\pi} \frac{dI_\nu}{ds} d\omega,$$

where ω is solid angle, and we have used the definition of the spectral flux, $\mathbf{F}_\nu$. Substituting for dI_ν/ds from the radiative transfer equation (1), and assuming azimuthal symmetry yields:

$$Q_\nu = 2\pi k_\nu^m \rho_a \int_{-1}^{1} (I_\nu - S_\nu) d\mu' \tag{4}$$

$$\mu' = \cos\theta,$$

where the sign of μ' is indicated explicitly:

$$Q_\nu = 2\pi k_\nu^m \rho_a \left(\int_o^1 I_\nu^- d\mu + \int_o^1 I_\nu^+ d\mu - \int_{-1}^{1} S_\nu d\mu' \right). \tag{5}$$

Assuming that there is no scattering, k_ν^m may be replaced by κ_ν^m, the mass absorption coefficient. Assuming that all sources are thermal in nature, we have $S_\nu = B_\nu(T)$, where $B_\nu(T)$ is the Planck function. We assume that the temperature is a function of a single position variable and time only. Substituting S_ν and (2) into (5), and performing the integration in the third term, whose integrand is independent of direction, yields

$$Q_\nu = 2\pi \kappa_\nu^m \rho_a \left[\int_o^1 \left(\int_{-\infty}^{\tau_\nu} d\tau_\nu' B_\nu(\tau_\nu') e^{-(\tau_\nu - \tau_\nu')/\mu} + \right.\right.$$

$$\left.\left. \int_{\tau_\nu}^{\infty} d\tau_\nu' B_\nu(\tau_\nu') e^{-(\tau_\nu' - \tau_\nu)/\mu} \right) \frac{d\mu}{\mu} - 2B_\nu(\tau_\nu) \right]. \tag{6}$$

Assuming the grey case, $\kappa_\nu^m \equiv \kappa$ permits immediate integration of this equation over all frequencies. We define the total heating contributed by all frequencies, and write it in terms of the change in temperature,

$$\int_o^\infty Q_\nu d\nu \equiv Q = \rho_t C_p \frac{dT}{dt}(\tau,\ t),$$

where ρ_t is the total gas density. C_p is the specific heat at constant pressure, and is more appropriate than the specific heat at constant volume because the volume of an unconstrained gas parcel will change as the temperature changes. At any rate, this effect is small in the linear approximation used, so we will ignore any adiabatic temperature change effects. The Planck function integrated over all frequencies is $B = \sigma_B T^4/\pi$, where σ_B is the Stefan-Boltzmann constant, so (6) becomes

$$\frac{dT}{dt} = \frac{4\kappa\rho_a\sigma_B}{\rho_t C_p}\left[\frac{1}{2}\int_o^1\left(\int_{-\infty}^{\tau} d\tau' T^4 e^{-(\tau-\tau')/\mu} + \int_{\tau}^{\infty} d\tau' T^4 e^{-(\tau'-\tau)/\mu}\right)\frac{d\mu}{\mu} - T^4\right].$$

The next step is to linearize the problem by assuming a small perturbation to the isothermal background state, $T(\tau, t) = \overline{T} + \Theta(\tau, t)$. Keeping only background terms, and terms linear in the perturbation yields

$$\frac{d\Theta}{dt} \cong \frac{4\kappa\rho_a\sigma_B}{\rho_t C_p}\left[\frac{1}{2}\int_o^1\left(\int_{-\infty}^{\tau} d\tau'(\overline{T}^4 + 4\overline{T}^3\Theta)e^{-(\tau-\tau')\mu} + \right.\right.$$

$$\left.\left.\int_{\tau}^{\infty} d\tau'(\overline{T}^4 + 4\overline{T}^3\Theta)e^{-(\tau'-\tau)/\mu}\right)\frac{d\mu}{\mu} - (\overline{T}^4 + 4\overline{T}^3\Theta)\right].$$

Here we have used Spiegel's result that the effect of the temperature perturbation through the temperature dependence of the mass absorption coefficient, κ, is zero to first order (Spiegel, 1957). That is, the heating/cooling due to the change in extinction approximately cancels the change in emissive cooling/heating. Note that Θ can be positive or negative.

The background state is constant, so the emission of the background atmosphere just balances its absorption. Thus the terms describing the background radiative equilibrium state drop out of the above equation. This leaves us with the equation governing the temporal and spatial evolution of the temperature perturbation;

$$\frac{d\Theta}{dt} = \gamma\left[\frac{1}{2}\int_o^1\left(\int_{-\infty}^{\tau} d\tau'\Theta e^{-(\tau-\tau')/\mu} + \int_{\tau}^{\infty} d\tau'\Theta e^{-(\tau'-\tau)/\mu}\right)\frac{d\mu}{\mu} - \Theta\right] \tag{7}$$

$$\gamma \equiv \frac{16\sigma_B\overline{T}^3\kappa\rho_a}{C_p\rho_t}. \tag{8}$$

3. Solution and limiting forms

a. General

In an infinite, homogeneous medium this equation can be solved by separation of variables, $\Theta(\tau, t) \equiv \theta(\tau)\phi(t)$. The usual method of substitution into the equation, followed by division by $\theta\phi$ and setting each side equal to a separation constant $-n$, yields a simple equation for ϕ, which has the solution

$$\phi(t) = \phi(t=0)e^{-nt}, \tag{9}$$

where n is given by the second equation resulting from the separation:

$$n = \gamma - \frac{\gamma}{2\theta(\tau)}\int_o^1\left(\int_{-\infty}^{\tau} d\tau'\theta(\tau')e^{-(\tau-\tau')/\mu} + \int_{\tau}^{\infty} d\tau'\theta(\tau')e^{-(\tau'-\tau)/\mu}\right)\frac{d\mu}{\mu}. \tag{10}$$

The first step of the Fourier analysis is to assume spatial solutions of the form $\theta(\tau) = e^{ik\tau}$. The actual solution will be the real part of the solution produced by this procedure. The quantity k is dimensionless, as a result of τ being dimensionless. Solutions of this form also make the decay rate of the solution dependent on the wavelength of the decaying (sinusoidal) perturbation. The decay rate for each component of the perturbation is then given by

$$n(k) = \gamma - \frac{\gamma}{2}\int_0^1 \left(\int_{-\infty}^{\tau} d\tau' e^{(ik+\frac{1}{\mu})(\tau'-\tau)} \int_{\tau}^{\infty} d\tau' e^{(ik-\frac{1}{\mu})(\tau'-\tau)}\right)\frac{d\mu}{\mu}.$$

The integrations over τ', and then μ, are simple to perform and the imaginary parts cancel, so we are left with

$$n(k) = \gamma\left(1 - \frac{1}{k}\tan^{-1} k\right)$$

$$= \gamma\left(1 - \frac{1}{k}\cot^{-1}\frac{1}{k}\right), \quad (11)$$

which is Spiegel's result, except with a nondimensional wavenumber (Spiegel had k/κ instead of k as the argument). As he pointed out, this function increases monotonically from zero to γ as k increases from zero to infinity, as can be seen in Figure 1. We will examine its behavior more carefully below.

The $e^{ik\tau}$ form a complete set of linearly independent eigenfunctions, in terms of which any well behaved function of τ can be expanded. In general we can write down the solution for a single sinusoidal component as $\Theta_k(\tau, t) = R_e\phi(0)e^{-n(k)t}e^{ik\tau}$. In particular we can choose an initial

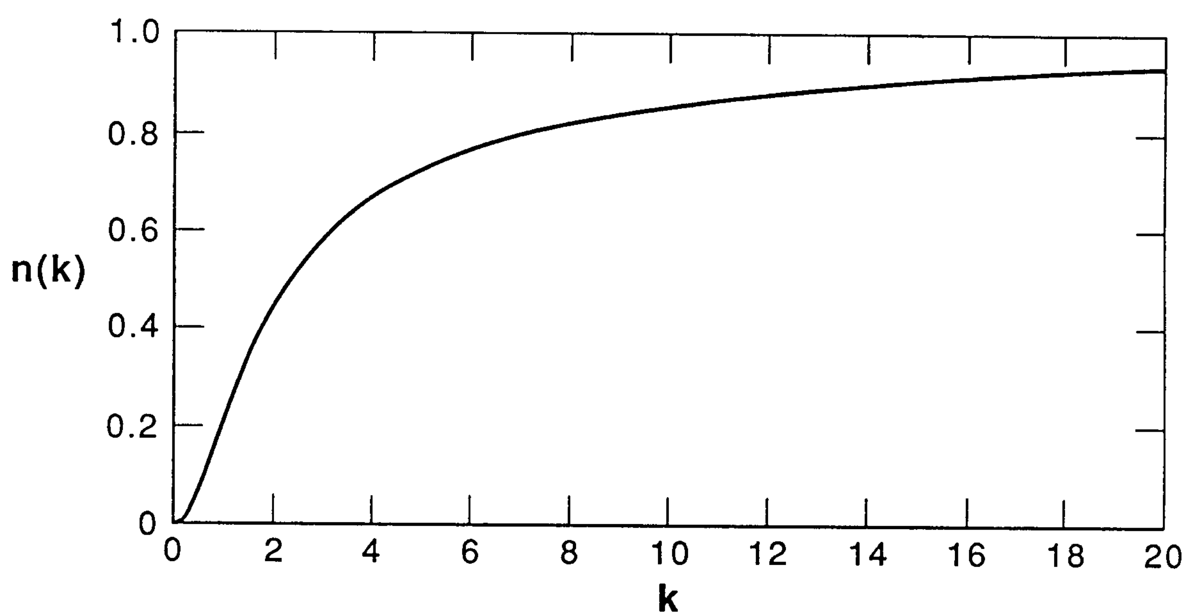

Fig. 1. The decay rate $n(k)$ in units of γ.

temperature distribution and find its evolution in time by adding up all of its (time dependent) Fourier components, properly weighted by its Fourier transform;

$$\Theta(\tau,\ t) = Re \int_o^\infty F(k)\phi(0)e^{-n(k)t}e^{ik\tau}dk$$

$$= \phi(0)\int_o^\infty F(k)\cos(k\tau)\exp\Big(-\gamma t\Big(1-\frac{1}{k}\tan^{-1}k\Big)\Big)dk, \tag{12}$$

where $F(k)$ is the Fourier transform of $\Theta(\tau;\ t=0)$. The evaluation of (12) is difficult for an arbitrary temperature distribution. However, the exponential factor can be approximated quite easily in both the optically thin and thick limits, and with a judicious choice of temperature distribution we can obtain closed form solutions.

A particularly simple, but realistic initial temperature perturbation is known as the "Witch of Agnesi". It has the form $\Theta(\tau,\ 0) = \phi(0)/(1+(\tau/a)^2)$, which is a bell-shaped curve with a denoting the half-width at half-maximum. The single maximum is at $\tau = 0$, and the amplitude falls off as $(a/\tau)^2$ for larger τ. Besides it's "realistic" shape, it also has the attraction of having a simple, exponential Fourier transform $F(k) = ae^{-ak}$. However this still does not yield an analytic result, and we must resort to approximating the decay rate $n(k)$. As Spiegel noted, this decay rate behaves approximately as k^2 for small k, which is the optically thick limit, while in the optically thin limit $n(k)$ approaches γ, and becomes independent of k.

b. Optically thin limit

We will examine the optically thin limit first, in which the half width of the perturbation is much less than one optical depth. Such a distribution is dominated by the larger wavenumber components of the Fourier spectrum, which permits us to only approximate the decay rate for large k. Hence it will be sufficient to approximate $n(k)$ by a finite number of linear segments, the number depending on the desired accuracy of the results and the actual width of the perturbation.

The approximation for $n(k)$ can be succinctly written as

$$n(k) \cong \Big\{\gamma(m_i k + n_i) \text{ for } k_i \le k \le k_{i+1},\quad i = 1,\ldots,\ell\Big\}, \tag{13}$$

where m_i is the slope, n_i is the "n-intercept,", k_i and k_{i+1} are the domain endpoints of the i^{th} segment, and ℓ is the number of segments in the approximation. We can then write equation (13) in the form

$$\Theta \cong \phi(0)a\sum_1^\ell \int_{k_i}^{k_{i+1}} \cos(k\tau)e^{-ak-\gamma t(m_i k+n_i)}dk. \tag{14}$$

The first segment must pass through $n = 0$ at $k = 0$ because $k = 0$ represents the isothermal component of the perturbation, which cannot evolve with time in an infinite atmosphere. We can make the last "segment" a horizontal half-line starting at k_ℓ and going out to infinity, since n

approaches γ for large k. Thus the last integral of (14) is an improper one, and when the limit of $k_{\ell+1} \to \infty$ is taken its value becomes the value of the integral at k_ℓ. Each integral in (14) can be evaluated using the same formula (Gradshteyn and Rhyzhik 2.662 # 2).

$$\int e^{ax} \cos bx \; dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx).$$

Gathering terms in cos $k_i \tau$ and sin $k_i \tau$ yields

$$\Theta(\tau,\; t) \cong \frac{a\phi(0)(a + m_1 \gamma t)}{(a + m_1 \gamma t)^2 + \tau^2} +$$

$$a\phi(0) \sum_{i=2}^{\ell} \Big\{ (E_1 - E_2) \tau e^{-ak_i} \sin k_i \tau +$$

$$[(a + m_i \gamma t) E_2 - (a + m_{i-1} \gamma t) E_1] e^{-ak_i} \cos k_i \tau \Big\}, \tag{15}$$

$$E_1(t,\; k_i) \equiv \frac{\exp[-(m_{i-1} k_i + n_{i-1}) \gamma t]}{(a + m_{i-1}) \gamma t)^2 + \tau^2},$$

$$E_2(t,\; k_i) \equiv \frac{\exp[-(m_i k_i + n_i) \gamma t]}{(a + m_i \gamma t)^2 + \tau^2}.$$

This is not a particularly transparent formula, but some features of its behavior can be easily seen. When evaluated at $t = 0$ the terms in the sum disappear and we are left with the initial temperature perturbation, as required. If, instead, we examine the solution at $\tau = 0$ as a function of time, we see that the coefficient of the cosine expression in the sum consists of two parts. Each of these parts decays with time, but the difference between them increases with time because $n(k)$ is an increasing function of k. That is, each segment represents larger values of the decay rate $n(k)$ than the one preceding it. Thus, E_2 decays faster than E_1, and the net effect is to reduce the amplitude of the perturbation monotonically with time. This behavior characterizes regions near the peak, while the temperature at outlying regions will initially increase with time as the positive contribution from the sine term overcomes the negative contribution from the cosine term. Lastly, for very large times the perturbation decays to zero amplitude, as can be seen by inspection.

Some examples of the evolution of the perturbation for various half-widths can be seen in Figures 2a-c. They use the approximation for the decay rate given by (13) and the example data given in Table 1, below. The behavior of the solution is what we expect intuitively; the initial decay of the peak is relatively rapid, as the large-wavenumber components with their relatively rapid decay rates disappear. The remaining, small-wavenumber, components give the distribution its broadened appearance. It is noteworthy that despite the small number of segments (three) used in the approximation of $n(k)$, and hence the small number of wavenumbers which appear in (14), the figures are distinctly lacking in the waviness we might expect.

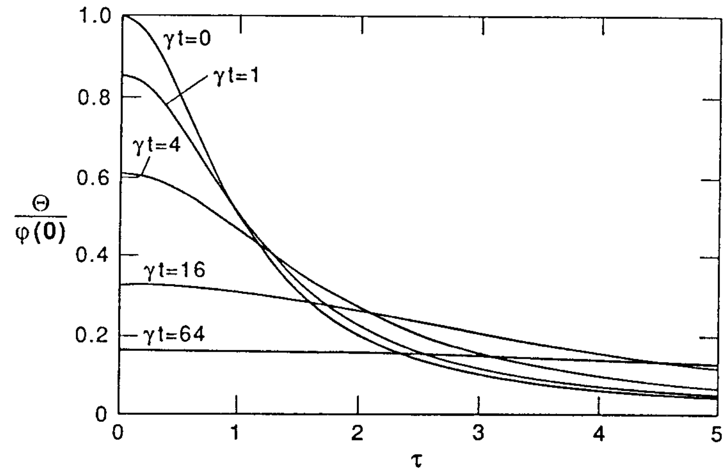

Figure 2a

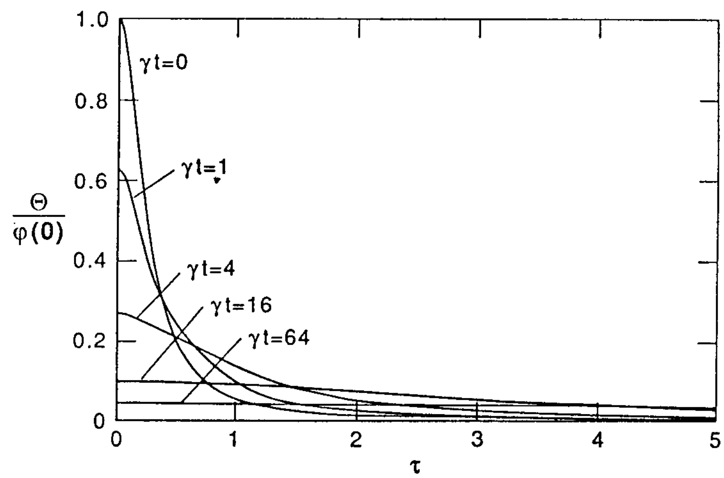

Figure 2b

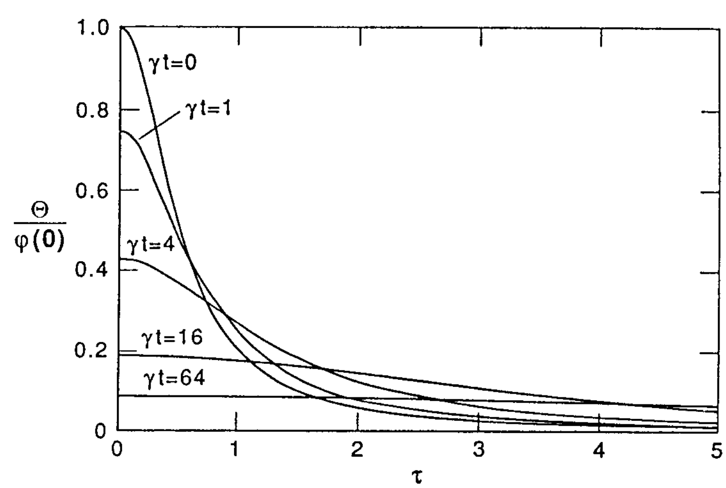

Figure 2c

Fig. 2a-c. Temporal evolution of non-dimensional perturbation $\Theta/\phi(0)$ with half-widths of optical depths a) 1.0, b) 0.5, c) 0.25, using the optically thin description (Eqs. 15). Each curve denotes the shape of the perturbation at the dimensionless time γt.

Table 1. Values for the limits, slopes, and n-intercepts of the line segment approximation of Spiegel's decay function $n(k)$ used for Figure 2a-c

i	k_i	m_i	n_i
1	0.0	0.05	0.0
2	0.15	0.2	-0.027
3	4.27	0.0037	0.812
4	51.0	0.0	1.0

c. The optically thick limit

When the half-width is much greater than unity the initial perturbation is described primarily by the small wavenumber Fourier components, and Spiegel's function can be approximated by

$$n(k) \approx \gamma\left(1 - \frac{1}{k}\left(k - \frac{1}{3}k^3 + \ldots\right)\right)$$

$$\approx \frac{\gamma}{3}k^2 \tag{16}$$

We will see in Appendix A that this result is identical to that of the Eddington approximation. In our case the factor of 1/3 is a result of the approximation of $\tan^{-1}k$ for small k. Spiegel identifies the factor of $\gamma/3(k\rho_a)^2$, which arises from writing k in dimensional form in the above approximation, as a thermal diffusivity. This is appropriate because in the optically thick limit, and in the grey case, radiative transfer can be described as a diffusion process; photons must travel many mean free paths before escaping to the unperturbed regions. The process proceeds differentially, so the problem can be reformulated as a differential equation. This alternative derivation is pursued in Appendix A.

In either formulation the problem reduces to evaluating (12) with $n(k)$ approximated by $\gamma k^2/3$. With this approximation the expression for the evolution of the optically-thick perturbation becomes

$$\Theta(\tau,\ t) \approx \phi(0)a\int_o^\infty e^{-\frac{\gamma t k^2}{3} - ak}\cos k\tau dk \tag{17}$$

There are two readily available solutions to this integral. The first is from Gradshteyn and Rhysik (3.897), and involves the "probability integral" $\Phi(z)$.

$$\Theta(\tau,\ t) = \frac{\phi(0)a}{4}\sqrt{\frac{3\pi}{\gamma t}}\{e^{z^*}[1 - \Phi(z^*)] + e^{z}[1 - \Phi(z)]\}, \tag{18}$$

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_o^z e^{-t^2}dt.$$

$$z \equiv \frac{a + i\tau}{\sqrt{4\gamma t/3}}.$$

A somewhat simpler form can be found in Abramowitz and Stegun (7.4.2).

$$\Theta(\tau,\ t) = \phi(0)a\frac{1}{2}\sqrt{\frac{3\pi}{\gamma t}}\mathrm{Re}\ e^{z^{*2}}\operatorname{erfc}(z^*), \tag{19}$$

where z^* denotes the complex conjugate of z.

4. Energy density

Our use of a line segment to approximate the quadratic region of the decay rate spectrum should lead us to suspect that the initial amplitudes of the small wavenumbers are overestimated, and hence the long wavelength components are overestimated at large times. In other words the temperatures in Figures 2a-c for large time are too large. To estimate how much error there actually is in this numerical solution we may calculate how much energy is in the initial perturbation, assume that energy is conserved, and then compare the energy in both thermal and radiative forms due to the perturbation at later times with this initial amount. The perturbation thermal energy of the gas $U_g(t)$ can be found from the temperature perturbation, $\Theta(\tau, t)$,

$$U_g(t) = \rho_t C_p \int_{-\infty}^{\infty} \Theta(\tau,\ t)d\tau. \tag{20}$$

The perturbation radiative energy requires more effort. The spectral energy density is defined as

$$u_\nu = \frac{1}{c}\int_{4\pi} I_\nu d\omega = \frac{2\pi}{c}\int_{-1}^{1} I_\nu d\mu,$$

where c is the speed of light. We can substitute for the integral of I_ν from (1) and then use the assumption of only thermal sources to substitute B_ν for S_ν and use the grey case assumption to substitute κ for k_ν^m;

$$u_\nu = \frac{Q_\nu}{\rho_a \kappa c} = \frac{1}{c}\int_{-1}^{1} B_\nu d\mu.$$

After integrating over all frequencies to get the total radiative energy density, we can write both Q and B in terms of the temperature

$$u_r = \frac{\rho_t C_p}{\rho_a \kappa c}\frac{\partial T}{\partial t} + \frac{2\sigma_B}{c\pi}T^4.$$

Linearizing this equation as before, and noting that the time-independent, background radiation energy terms cancel, we obtain the perturbation radiation energy density as a function of position and time.

$$u_r'(\tau,\ t) = \frac{\rho_t C_p}{\rho_a \kappa c}\frac{\partial \Theta(\tau,\ t)}{\partial t} + \frac{2\sigma_B 4\overline{T}^3}{c\pi}\Theta(\tau,\ t). \tag{21}$$

Integrating (21) over the whole medium and adding the perturbation thermal energy (20) yields the total perturbation energy in terms of the temperature perturbation only, as a function of time,

$$U_p(t) = \frac{\rho_t C_p}{\rho_a \kappa c} \int_{-\infty}^{\infty} \frac{\partial \Theta(\tau,\ t)}{\partial t} d\tau + \frac{8\sigma_B \overline{T}^3}{\pi c} \int_{-\infty}^{\infty} \Theta(\tau,\ t) d\tau + \rho_t C_p \int_{-\infty}^{\infty} \Theta(\tau,\ t) d\tau,$$

which can be written more compactly as

$$U_p(t) = \frac{\rho_t C_p}{\rho_a \kappa c} \left\{ \int_{-\infty}^{\infty} \frac{\partial \Theta(\tau,\ t)}{\partial t} d\tau + \left[\frac{\gamma}{2\pi} + \rho_a \kappa c \right] \int_{-\infty}^{\infty} \Theta(\tau,\ t) d\tau \right\}. \qquad (22)$$

In the non-dissipative case treated here the total perturbation energy should remain constant, and equal to the initial perturbation energy. In principle the perturbation energy at all times can be evaluated using the approximate expression for Θ given by (15) or (18) in (22). Its departure from the initial constant value is a measure of the overall error of the numerical solution. Unfortunately the expression for the initial perturbation is time independent, which means that one of the (time dependent) approximate solutions, (15) or (18), must be used in (22) to find the initial energy. Thus even the initial energy will only be approximately correct, and must be evaluated numerically, which means that this method will provide only a relative check on the accuracy of the solutions (15) and (18).

5. Summary and conclusions

We have evaluated the initial-value problem of a one dimensional temperature "pulse" decaying radiatively in a grey, infinite, homogeneous medium. We have presented numerical solutions in two cases; that of the optically-thin and optically-thick limits. The results confirm our intuition that the perturbation relaxes to the radiative equilibrium state by reduction of its amplitude, and by gradual broadening of its width. At the same time, the relaxation time gradually increases, with increasing spatial width. We have also shown how, in principle, the conservation of energy may be used as a partial check on the numerical accuracy of the solution.

More realistic atmospheres and other temperature perturbations could be treated by evaluating the closed-form solution (12) with modern numerical methods for determining inverse Fourier transforms. Making the atmospheres non-isothermal, or perhaps bounded, but retaining the radiative equilibrium, might be the easiest complexity to include. It would also be possible to treat the non-grey case using some relatively simple line profile.

6. Acknowledgements

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APPENDIX A

An alternative development for the optically thick cases, when the half-width is significantly greater than optical depth unity, begins with the radiative transfer equation for an atmosphere with no scattering, and only thermal sources;

$$\frac{\partial I_\nu}{\partial s} = -\kappa_\nu^m \rho_a (I_\nu - B_\nu),$$

where s is distance. The vertical optical depth will be dfined by $\tau_{vert}^m = \mu \kappa_\nu^m \rho_a s$, and in an optically thick atmosphere $\partial I_\nu / \partial \tau \cong \partial B_\nu / \partial \tau$. Thus the radiative transfer equation becomes

$$-I_\nu \cong \mu \frac{\partial B_\nu}{\partial \tau} - B_\nu. \qquad (A1)$$

Integration over solid angle gives the net spectral flux from the first term on the right hand side of $(A1)$ which has a component only in the τ direction (use the scalar F_ν to denote this component). The second term on the right disappears because it is isotropic, which leaves

$$-F_\nu = 2\pi \int_{-1}^{1} \frac{\partial B_\nu}{\partial \tau_\nu} \mu d\mu.$$

Taking the divergence yields the spectral heating rate;

$$Q_\nu \equiv -\nabla \cdot \mathbf{F}_\nu = -\kappa_\nu \rho_a \mu \frac{\partial F_\nu}{\partial \tau_\nu},$$

$$= 2\pi \kappa_\nu \rho_a \int_{-1}^{1} \frac{\partial^2 B_\nu}{\partial \tau^2} \mu^2 d\mu.$$

B_ν is isotropic, so the integration over μ can be performed, yielding the familiar factor of 1/3, which is the Eddington approximation (The average value of the squared cosine of the polar angle).

As before we integrate over all frequencies to get the total heating rate, Q, which is equal to $\rho_t C_p \partial T / \partial t$. Thus

$$\rho_t C_p \frac{\partial T}{\partial t} = \frac{4\pi \kappa \rho_a}{3} \frac{\partial^2 B}{\partial \tau^2}.$$

Again, $B = \sigma_B T^4 / \pi$, and we linearize the equation by assuming an isothermal background with a small perturbation, and keeping only terms linear in the perturbation. The equation for the evolution of the perturbation $\Theta(\tau, t)$ is then

$$\frac{\partial \Theta}{\partial t} = \frac{4\kappa \rho_a \sigma_B 4\overline{T}^3}{3\rho_t C_p} \frac{\partial^2 \Theta}{\partial \tau^2}.$$

This also can be solved by separation of variables, using $-n$ as the separation constant and the component functions $\Theta = \phi(t)\chi(\tau)$. The time dependence is given by the same function $\phi(t) = \phi(0)e^{-nt}$, but this time n is given by a differential equation;

$$\frac{\gamma}{3} \frac{d^2 \chi}{d\tau^2} = -n\chi.$$

Putting this in standard form leads us to look for solutions in the form of Fourier components; $\chi = e^{ik\tau}$, where k is defined by $n(k) = \frac{\gamma}{3}k^2$, which will be recognized as the small k approximation of Spiegel's $n(k)$, equation (11). The total solution is then given by the real part of the Fourier integral

$$\Theta(\tau,\ t) = Re \int_o^\infty \phi(0) a e^{-ka} e^{ik\tau} e^{-\frac{\gamma t k^2}{3}} dk$$

$$= a\phi(0) \int_o^\infty \cos k\tau e^{-ak-\frac{\gamma t k^2}{3}} dk,$$

where we can take the integral out to infinity because in the optically thick limit the perturbation has very small contributions from the large k values. This can be seen clearly in the Fourier transform of the perturbation, which decays exponentially with k, with a "rate" a. This integral is identical to (17), which is discussed in the body of the paper.

REFERENCES

Abramowitz, M., and I. Stegun, 1965. Handbook of Mathematical Functions. Dover Publications.

Gay, C., 1978. Dissipation of Temperature Perturbations in a Planetary Atmosphere by Radiative Transfer, Ph. D. Thesis, University of Colorado.

Gay, C., and G. E. Thomas, 1980. Radiative temperature dissipation in a finite atmosphere I. The homogeneous case. *Quant. Spectrosc. Radiat. Transfer,* **25**, 351-380.

Gay, C., and G. E. Thomas, 1982. Radiative equilibrium in the middle atmosphere, *Geofis. Int.*, **21**, 69-80.

Gradshteyn, I. S., and I. M. Rhyzhik, 1980. Tables of Integrals, Series, and Products. Academic Press, Fourth Ed.

Nagirner, D. I., 1979. Calculation of the spectrum of the integral equation radiative transfer II: Plane layer of finite optical thickness, *Astrofizika,* **15**, 485-495 (English translation; Plenum Publ. Co., Astrofizika, **15**, 324-330, 1980).

Sasamori, T., and J. London, 1966. The decay of small temperature perturbations by thermal radiation in the atmosphere. *J. Atmos. Sci.*, **23**, 543-554.

Spiegel, E. A., 1957. The smoothing of temperature fluctuations by radiative transfer. *Astrophys. J.,* **126**, 202-207.