Atmósfera (1992), **5**, *pp. 135-155*

Comparisons of low-order atmospheric dynamic systems

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(Manuscript received Aug. 15, 1991; accepted in final form Nov. 25, 1991)

RESUMEN

Un modelo de dos niveles de orden bajo, casi-geostrófico, se utiliza para investigar la respuesta a calentamiento externo. El calentamiento tiene variaciones en las direcciones norte-sur y este-oeste. La disipación friccional (por fricción) se incorpora usando tanto la fricción interna como en la capa límite. El flujo zonal se describe mediante las variables dependientes, una para el flujo medio vertical y la otra para el de cizalladura vertical. Las restantes cuatro variables dependientes en el modelo son las amplitudes de las componentes seno y coseno de una onda viajera en el flujo medio vertical y en el de cizalladura.

El modelo de orden bajo es estable en el sentido de que las trayectorias que empiezan fuera de un cierto círculo cruzan éste aproximándose al origen del espacio de seis dimensiones. El modelo también tiene la propiedad de que la razón de cambio de un pequeño volumen es negativa, indicando que éste se reducirá a cero, entonces cualquier atractor que pueda existir es de volumen cero. Un estudio detallado de los múltiples estados estacionarios del modelo y de su estabilidad se pospone para un trabajo posterior. En este estudio nos basamos en un número de integraciones numéricas a largo plazo que muestran que el modelo tiende a una solución estable o a una estacionaria, o bien a una solución periódica dependiente del tiempo. Parece entonces que el modelo no contiene soluciones caóticas.

Se realizan comparaciones con el modelo de tres parámetros recientemente publicado por Lorenz y con los modelos desarrollados por Saltzman *et al.* Este modelo de Lorez que contiene soluciones caóticas para un forzamiento externo norte-sur suficientemente grande, puede obtenerse como un caso especial del modelo de seis parámetros. El comportamiento distinto de los dos modelos puede explicarse, por las suposiciones que son necesarias considerar para obtener a partir del modelo complicado, el más simple. Se demuestra que el modelo de Lorenz descubre el flujo térmico del modelo de dos niveles, y que la diferencia de fase entre el flujo térmico y el flujo medio de las ondas en el modelo, siempre es de un cuarto de la longitud de onda asegurando con esto que el transporte sur-norte de calor sensible es un máximo para ciertas amplitudes. También se señala que se requieren valores grandes del calentamiento externo para obtener caos en el modelo de Lorenz. Estos resultados también se encuentran en el modelo de Saltzman que es una generalización del modelo de Lorenz aunque ambos tiene tres variables dependientes, solamente.

Parece entonces que para explicar las variaciones inter-anuales de la atmósfera en términos de comportamiento caótico en la estación fría y no caótica en la estación caliente, requerirá investigaciones posteriores usando modelos que puedan simular los procesos en cascada que se dan en la atmósfera real.

ABSTRACT

A low-order, quasi-geostrophic, two-level model is used to investigate the response to external heating. The heating has variations in both south-north and west-east directions. The frictional dissipation is incorporated by using both boundary layer and internal friction. The zonal flow is described by two dependent variables, one for the vertical mean flow and the other for the vertical shear flow. The remaining four dependent variables in the model are the amplitudes of the sine- and cosine-components of a travelling wave in the vertical mean flow and in the vertical shear flow.

The low-order model is stable in the sense that trajectories starting outside a certain circle will cross the circle and approach the origin of the six-dimensional space. The model has also the property that the rate of change of a small volume is negative indicating that the small volume will shrink to zero. Any attractor, which may exist, is thus of zero volume. A detailed study of the multiple steady states of the model and their stability is postponed to a later publication. In this study we rely on a number of long-term numerical integrations, which show that the model approaches either a stable, steady state or a periodical time-dependent solution. It appears therefore that the model does not contain chaotic solutions.

Comparisons are made with the three parameter model recently published by Lorenz and with the models developed by Saltzman *et al.* This Lorenz model, which contain chaotic solutions for sufficiently large south-north external forcing, can be obtained as a special case of the six parameter model. The different behavior of the two models may be explained by the assumptions, which are necessary to obtain the simpler model from the other. It is shown that the Lorenz-model describes the thermal flow of the two-level model, and that the phase difference between the thermal and the mean model flow waves always is a quarter of the wavelength assuring that the south-north transport of sensible heat is at a maximum for given amplitudes. It is also pointed out that large values of the external heating are necessary to obtain chaos in the Lorenz-model. These results are also found in the Saltzman-model, which is a generalization of the Lorenz-model, although both have three dependent variables only. It appears therefore that to explain the inter-annual variations of the atmosphere in terms of chaotic behavior in the cold season and non-chaotic behavior in the warm season will require further investigations using models, which can simulate the cascade processes in the real atmosphere.

1. Introduction

Low order models of the atmospheric general circulation can be used to understand some aspects of the mechanisms at work in the climate system. Thompson (1987) studied a two-level quasi-geostrophic model with differential heating in the meridional plane and with lateral diffusion as the dissipating mechanism. The author (Wiin-Nielsen, 1991, hereafter referred to as A) has recently studied the response to the external heating of the atmosphere by using a low order system, which has six components. The system is based on the quasi-geostrophic model and has two components to describe the zonal structure, while the remaining four components are used to describe the waves. The main result of the study is that for each intensity of the heating (constant in time) and each wavelength there exists one and only one stationary, stable state. Two kinds of stable steady states exist. The most important is a baroclinic wave, which slopes westward with heigth and thus transport sensible heat from south to north. It has the characteristic phase difference between the temperature field and the geopotential field giving warm air advection into the ridge of the height field and cold air advection into the trough. For a typical baroclinic wave in the steady state the phase differnece is of the order of 1/6 to 1/5 of wavelength. The other stable steady state, which exists outside the baroclinic domain has an equivalent barotropic structure.

The model used in A has an external heating which varies with latitude, but is independent of longitude. It has in other words been assumed that the main effect of the heating in the atmosphere is to create a temperature difference between the high and the low latitudes which will create the baroclinic waves by the baroclinic instability mechanism provided the horizontal temperature gradient is sufficiently large. This assumption is certainly convenient because it permits a transformation to a set of new equations expressed in terms of the heat transport, the eddy kinetic energies in the vertical mean field and the vertical shear field, and an additional transport quantity, analysed in A. On the other hand, the real atmosphere has longitudinal variations of the heating created partly for the long waves by the distributions of continents and oceans and partly for the Rossby waves by the atmospheric waves themselves. It may thus be of importance to include the longitudinal variation in the model. One of the purposes of this paper is to attempt to do this.

The analysis carried out in A concentrated on the steady states and their stability, but no numerical integrations were carried out. Lorenz (1984, 1990) has investigated the long term behavior of an even simpler model containing only three dependent variables. This model may be obtained as a simplification of the model described in the previous paragraph. In the most recent study Lorenz (1990) has demonstrated that for a sufficiently large external forcing his model has chaotic solutions, while periodic solutions are characteristic for smaller external forcing. These properties of the model are then used to investigate the interannual variations of the model atmosphere with the tentative conclusion that interannual variations may be described in terms of the chaotic behavior during the winter season, while non-chaotic, periodical behavior should be typical for the summer season. A second purpose of this study is to compare the simple Lorenz model with the slightly more general model used in the present study.

Saltzman *et al.* (1989) has made a study of an eight component model. Compared to the model in this paper it has an additional component in the zonal flow at each of the two levels. The major part of their study is however concerned with a three component low order model, which in many ways is similar to the Lorenz-model, but also different in including important mechanisms, which are neglected in the model formulated by Lorenz. The models are similar in the reduction to a three component system. In particular, the assumptions concerning the wave structure, treated in detail in section 4 of this paper, are the same in the two models. On the other hand, the beta effect, which is neglected in the Lorenz-model, is retained in the three component model derived by Saltzman and his colleagues, who also, in an *ad doc manner*, include a specified poleward eddy momentum transport due to wave structures and wave-wave interactions not explicitly represented in the model. It would appear that the eddy momentum transport is stipulated in a too general form, since the eddy momentum transport must vanish at the lateral boundaries. When these conditions are incorporated in the prescribed momentum transport, it reduces to a simple trigonometric form, because the boundary conditions do not permit the remaining terms. This inconsistency has no influence on the results obtained for the three level model.

The generalized three component Saltzman-model has been investigated by computing steady states, and the stability of these states have been determined. In addition, the seasonal variations have been obtained by numerical integrations. In our study we shall have more limited goals as outlined below.

The generalization of the previous six component model to contain longitudinal variations of the heating complicates the mathematical analysis of the model equations. It is no longer possible to transform the equations in the same manner as in A. One can in principle determine all steady states by using the eddy equations to determine the amplitudes of the waves in terms of the longitudinal heating and the zonal flow and proceed to use the equations for the zonal flow to calculate all stationary states. This procedure requires a series of cumbersome algebraic manipulations, the evaluations of a number of matrices and the solution of a high degree algebraic equation. In view of this situation it was decided to rely on long term numerical integrations to demonstrate some properties of the system. However, a study of the steady states and their stability will be published separately in the near future. Since the equations for the system were discussed in A, we shall refer to this paper, but restate the equations here.

2. The model

The model will be the same as the one employed in A. This means a standard two-level, quasi-geostrophic model with the curl of the surface stress proportional to the relative vorticity at 100 kPa. The curl of the internal stress becomes proportional to the thermal vorticity. Since the two information levels are at 25 and 75 kPa we shall for simplicity take the vorticity at 100 kPa to be 1/2 of the vorticity at 75 kPa.

It will not be necessary to go in detail with respect to the equations. We shall adopt the following numerical values:

$$q^2 = \frac{2f_o^2}{\sigma P^2} = 4 \times 10^{-12} m^{-2}$$

$$\epsilon = 4 \times 10^{-6} s^{-1}$$

$$\epsilon_T = 1.2 \times 10^{-6} s^{-1}$$

$$\kappa = R/c_p = 0.286$$

$$f_o = 10^{-4} S^{-1}$$

$$\beta = 1.6 \times 10^{-11} m^{-1} s^{-1} \tag{2.1}$$

For the low order model we define the two streamfunctions by the following expressions:

$$\psi_* = \frac{B_*}{2\lambda}\sin(2\lambda y) + \frac{E_*}{k}\sin(\lambda y)\sin(kx) + \frac{F_*}{k}\sin(\lambda y)\cos(kx) \tag{2.2}$$

with an analogous expression for the thermal streamfunction in which the subcript $*$ is replaced by the subcript T. For the heating we adopt

$$Q = Q_z \sin(2\lambda y) + Q_s \sin(\lambda y)\sin(ky) + Q_c \sin(\lambda y)\cos(kx) \tag{2.3}$$

As the equation indicates we have introduced a dependence on both x and y in this specification. In A we had the first term only. In these expression $k = 2\pi/L$, where L is the wavelength, $\lambda = \pi/W$ where W is the width of the channel ($W = 10^7 m$), and the coefficients B, E and F have the dimension of velocity.

Following the same procedure as in A we derive the equations for the six component model. These equations are identical to those in A expect for an extra term in the eddy, thermal equations. We rewrite the equations here for easy reference introducing a short hand notation for some coefficients.

$$\frac{dB_*}{dt} = -\frac{\epsilon}{4}(B_* - B_T)$$

$$\frac{dB_T}{dt} = -a_o(E_*F_T - E_TF_*) + e(B_* - B_T) - e_TB_T + g_zQ_z$$

$$\frac{dE_*}{dt} = (a_* B_* - b_*)F_* + a_* B_T F_T - \frac{\epsilon}{4}(E_* - E_T)$$

$$\frac{dF_*}{dt} = -(a_* B_* - b_*)E_* - a_* B_T E_T - \frac{\epsilon}{4}(F_* - F_T) \tag{2.4}$$

$$\frac{dE_T}{dt} = (a_T B_* - b_T)F_T - c_T B_T F_* + \frac{\epsilon}{4}\frac{b_T}{b_*}(E_* - E_T) - \epsilon_T \frac{b_T}{b_*} E_T + g_e Q_s$$

$$\frac{dF_T}{dt} = -(a_T B_* - b_T)E_T - c_T B_T E_* + \frac{\epsilon}{4}\frac{b_T}{b_*}(F_* - F_T) - \epsilon_T \frac{b_T}{b_*} F_T + g_e Q_c$$

in which we have introduced the following notations:

$$a_* = k\frac{k^2 - 3\lambda^2}{2(\lambda^2 + k^2)}; \qquad b_* = \frac{\beta k}{\lambda^2 + k^2};$$

$$a_T = k\frac{k^2 + q^2 - 3\lambda^2}{2(\lambda^2 + k^2 + q^2)}; \quad b_T = \frac{\beta k}{\lambda^2 + k^2 + q^2}; \quad c_T = k\frac{k^2 - q^2 - 3\lambda^2}{2(\lambda^2 + k^2 + q^2)};$$

$$N = 1 + \frac{q^2}{4\lambda^2}; \qquad e = \frac{\epsilon}{4N}; \quad e_t = \frac{\epsilon_T}{N};$$

$$g_z = \frac{q^2 \kappa}{4\lambda f_o N}; \qquad g_e = \frac{k q^2 \kappa}{2(k^2 + \lambda^2 + q^2)}; \tag{2.5}$$

The quantity $(E_* F_T - E_T F_*)$, appearing in the second equation above is proportional to the transport of sensible heat as was shown in A.

If one were to pursue the same strategy as in A, i.e. to determine the steady states, it would be required to solve the last four equations in (2.4) for the four amplitudes. This could be done, since the equations are linear in the four quantities: E_*, F_*, E_T and F_T. They would be expressed in terms of the zonal parameters and the eddy heating given by Q_z and Q_c. From these expressions one would have to calculate the heat transport term in the second equation of (2.4). Using finally the condition in the steady state that $B_* = B_T$, one would end with an algebraic equation in one of the zonal variables.

For the moment we shall be satisfied with some examples of numerical integrations of the system stated above. We shall be interested in the nature of the solutions. In view of the contribution from Lorenz (1990) it is of importance to see if the system will produce chaotic solutions. The numerical integrations have been carried out using the Heun scheme for the integration in time. It proved sufficiently accurate for our purposes. We shall, however, start with some general considerations of the model equations.

We consider an infinitesimal volume element, V, in the six-dimensional space of the dependent variables. The logarithmic change of the volume in time is equal to the divergence of the "velocity" vector:

$$\frac{dV}{dt} = V\nabla \cdot (dB_*/dt,\ dB_T/dt;\ \epsilon_*/dt,\ dF_*/dt,\ \epsilon_T/dt,\ dF_T/dt) \tag{2.6}$$

From the equations we get:

$$\frac{dV}{dt} = -V\left(\epsilon + \epsilon_T + \frac{b_T}{2b_*}(\epsilon + 4\epsilon_T)\right) \tag{2.7}$$

Contrary to the system used by Lorenz (*loc. cit.*) it is thus seen that the small volume in time will be reduced to zero. If an attractor exists it will therefore have a zero volume.

We may also investigate the stability of the system as a whole. By this we understand that if we start the system in a point very far from the point (0, 0, 0, 0, 0, 0) in the six-dimensional space it will always behave in such a way that the trajectory eventually will end inside a finite volume. For this purpose we consider the total energy in the system. It consists of the sum of the available potential and the kinetic energy in the zonal flow and the wave. Denoting $P = 50$ hPa and g, the acceleration of gravity, we get:

$$E = \frac{P}{g}\left\{B_*^2/2 + NB_T^2/2 + \frac{\lambda^2 + k^2}{4k^2}(E_*^2 + F_*^2) + \frac{\lambda^2 + k^2 + q^2}{4k^2}(E_T^2 + F_T^2)\right\} \tag{2.8}$$

From (2.8) and the system of equations we may proceed to calculate the rate of change of the total energy. After some algebra of an elementary nature we find:

$$\frac{dE}{dt} = -\frac{P}{g}\{T1 + T2 + T3 + T4 + T5 + T6 - T7 - T8\} \tag{2.9}$$

in which the notations have the following meaning:

$$T1 = \frac{\epsilon}{4}(B_* - B_T)^2$$

$$T2 = \frac{\epsilon(\lambda^2 + k^2)}{8k^2}\{(E_* - E_T)^2 + (F_* - F_T)^2\}$$

$$T3 = \frac{q^2\kappa\overline{Q}}{4\lambda f_o}(B_T/2 - \hat{Q}_z)^2$$

$$T4 = \frac{q^2\kappa\overline{Q}}{2kf_o}\{(E_T/2 - \hat{Q}_s)^2 + (F_T/2 - \hat{Q}_c)^2\}$$

$$T5 = \left(\epsilon_T - \frac{q^2\kappa\overline{Q}}{16\lambda f_o}\right)B_T^2$$

$$T6 = \left\{\frac{\epsilon_T(\lambda^2 + k^2)}{k^2} - \frac{q^2\kappa\overline{Q}}{8kf_o}\right\}(E_T^2 + F_T^2)$$

$$T7 = \frac{q^2\kappa\overline{Q}}{4\lambda f_o}\hat{Q}_z^2$$

$$T8 = \frac{q^2\kappa\overline{Q}}{2kf_o}(\hat{Q}_s^2 + \hat{Q}_c^2)$$

In the above formula we have introduced a scaled value of the heating by writing

$$Q = \overline{Q}\hat{Q} \tag{2.10}$$

The purpose is to obtain a scaled value of Q, which in order of magnitude is comparable to the velocity components and a suitable value of $\overline{Q}$ is therefore $10^{-3}kJt^{-1}s^{-1}$. The right hand side of (2.9) is certainly negative, if the initial point on the trajectory in the six-dimensional space is far removed from the origin (0, 0, 0, 0, 0, 0). It can, however, be shown that dE/dt is positive in a region close to the origin. We may see this by investigating the sign of dE/dt in the point where $B_* = B_T = \hat{Q}_z$, $E_* = E_T = \hat{Q}_s$ and $F_* = F_T = \hat{Q}_c$. In this special case we find that the contribution to dE/dt from the zonal part is:

$$-\frac{P}{g}\left\{\epsilon_T - \frac{\overline{Q}q^2\kappa}{4\lambda f_o}\right\}\hat{Q}_z^2 \tag{2.11}$$

and it is easy to see that the contribution from this term is positive for the selected values of the parameters. The contributions from the eddy terms are:

$$-\frac{P}{g}\left\{\epsilon_T\frac{\lambda^2+k^2}{k^2} - \frac{\overline{Q}q^2\kappa}{2kf_o}\right\}(\hat{Q}_s^2 + \hat{Q}_c^2).$$

This contribution is wavelength dependent, but an evaluation of the parenthesis in the term for a wavelength corresponding to a typical baroclinic wave in the atmosphere assures us that the parenthesis is negative and the whole contribution therefore positive. We have thus shown that dE/dt is positive in a point of the six-dimensional space under consideration and thus by continuity in a small region around the point. We may thus conclude that the system is stable in the sense that its trajectory will remain in a finite region.

The main conclusion from this section is therefore that if the system has an attractor, it will be of vanishing volume, and that the system is stable in the sense that the trajectory will not go to infinity.

3. Some numerical integrations

In this section we shall describe some long-term integrations of the system described in section 2 of this paper. The integrations were carried out using the Heun scheme with a time step of 3 hours. Although we could have plotted the resulting values of the six parameters, we have preferred to concentrate on the waves, where the largest variability is found. It was decided to plot quantities, which are proportional to the eddy transport of the sensible heat and quantities proportional to the kinetic energy of the vertical mean flow and the vertical shear flow.

The first experiment simulates winter conditions. For this purpose we have kept Q_z and Q_c at constant values equal to $2.27 \times 10^{-2}kJt^{-1}s^{-1}$ and $4.5 \times 10^{-3}kJt^{-1}s^{-1}$ respectively, while $Q_s = 0$. These values are comparable with the values, obtained for the winter season, by Schaack, Johnson and Wei (1990) as typical for the troposphere. The integrations are carried out for a few years to avoid any influence of the initial conditions. Figure 1 shows a plot of the heat transport for a time period covering about 80 days at the end of the first year of integration. It indicates quite clearly periodic flow with a period close to 28 days. Figures 2 and 3 show the two kinetic energies for the same time period. Also these figures show periodic flow with the same period. Several other time integrations were perfomed with higher and lower values of the heating parameters. In all cases we have obtained periodic flows showing that the solutions are non-chaotic.

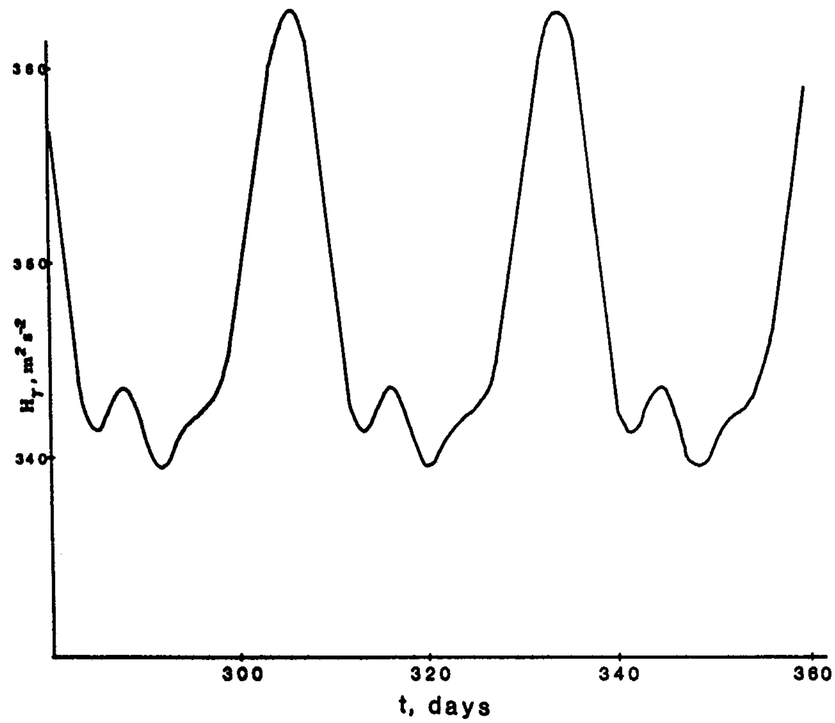

Fig. 1. A measure of the transport of sensible heat as a function of time showing two periods at the end of an integration over one year. $Q_s = 2.27 \times 10^{-2} kJt^{-1}s^{-1}$ and $Q_c = 4.5 \times 10^{-3} kJt^{-1}s^{-1}$.

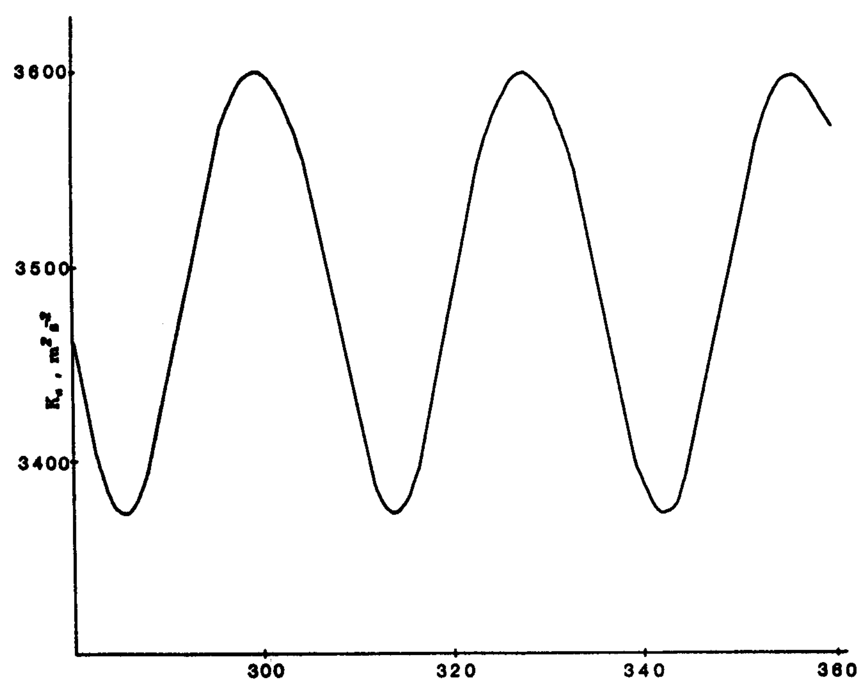

Fig. 2. The kinetic energy of the wave in the vertical mean flow. Parameters as in Figure 1.

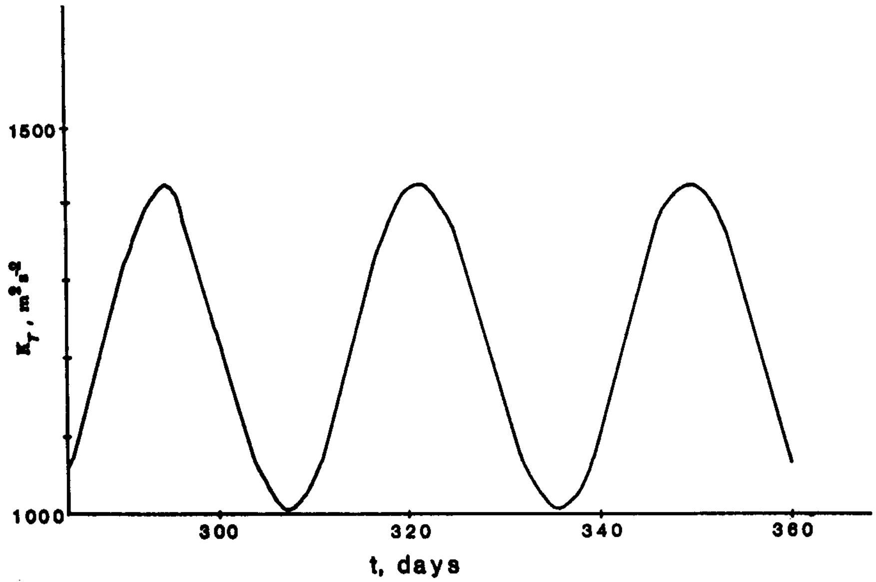

Fig. 3. The kinetic energy of the wave in the vertical shear flow. Parameters as in Figure 1.

It is of course also possible to simulate the annual variation in the model. The zonal part of the heating function was assumed to vary according to

$$Q_z(t) = 2.0 \times 10^{-2} + 2.84 \times 10^{-3} \cos(\omega t) \tag{3.1}$$

where ω is the frequency corresponding to the annual period. Q_C has the same constant value as before. We obtain then a solution, which in the first approximation may be characterized as a variation with the annual period, on which is superimposed a shorter period variation due to the baroclinic waves created by the instability of the current. A closer look shows also that there is a lag in the response in such a way that the waves of the largest amplitude appear some time after the beginning of the year, while the lowest level of the variability is found after the middle of the year. These statements are illustrated in Figure 4, which shows the shear flow kinetic energy as a function of time for one year after 5 years of integrations. Figure 5 shows the behavior of the same variable after 10 years of integration. Similar annual variations are found for the other variables.

All years are not identical. This is due to the fact that the time periods of one year does not contain an integer number of period for the waves. It follows that the state on a given date is not the same as the state of the model on the same date the previous and the following year. However, the main result is that the general nature of the state of the model does not change from one year to the next. The variability of the model is much more regular than is found from observations or in the model with only three dependent variables as designed by Lorenz (1990). The latter model shows an interannual variability, where the flow is chaotic in the wintertime and non-chaotic in the summertime for sufficiently large external forcing. Due to this fact the Lorenz-model is able to produce several types of flow during the summer season where some summers are characterized by very low amplitudes, while others display large amplitude flow. Due to this behavior Lorenz draws the tentative conclusion that interannual variability may be explained in terms of the chaotic behavior in the colder season.

The differences in behavior of the six and the three parameter models are at first sight surprising. Why is it that a slightly more general model with a few more degrees of freedom behaves

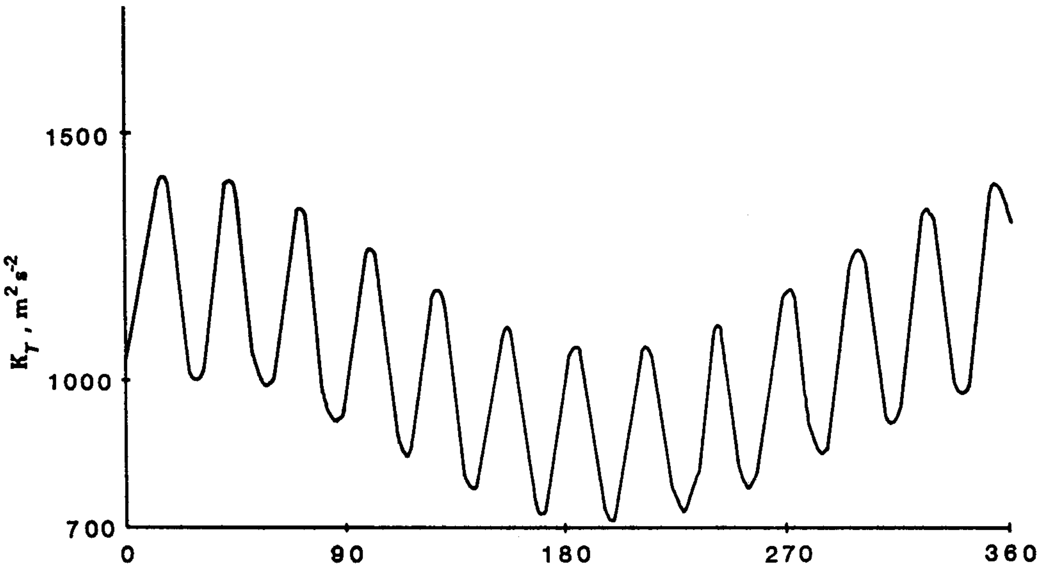

Fig. 4. The kinetic energy of the wave in the vertical shear flow for the sixth year in a simulation of the annual variation of the atmospheric flow. $Q_s = 2.0 \times 10^{-2} kJt^{-1}s^{-1}$, the amplitude of the annual variation of Q_s is $2.84 \times 10^{-3} kJt^{-1}s^{-1}$ and the constant value of Q_c is $4.5 \times 10^{-3} kJt^{-1}s^{-1}$.

in a much more regular way that the simpler model with only three dependent variables, which can be obtained as a special case of the more general model? Is it perhaps due to the way in which the smaller model is obtained as a special case of the other? The same questions can be raised in any investigation, in which low-order models are used to investigate a certain nonlinear

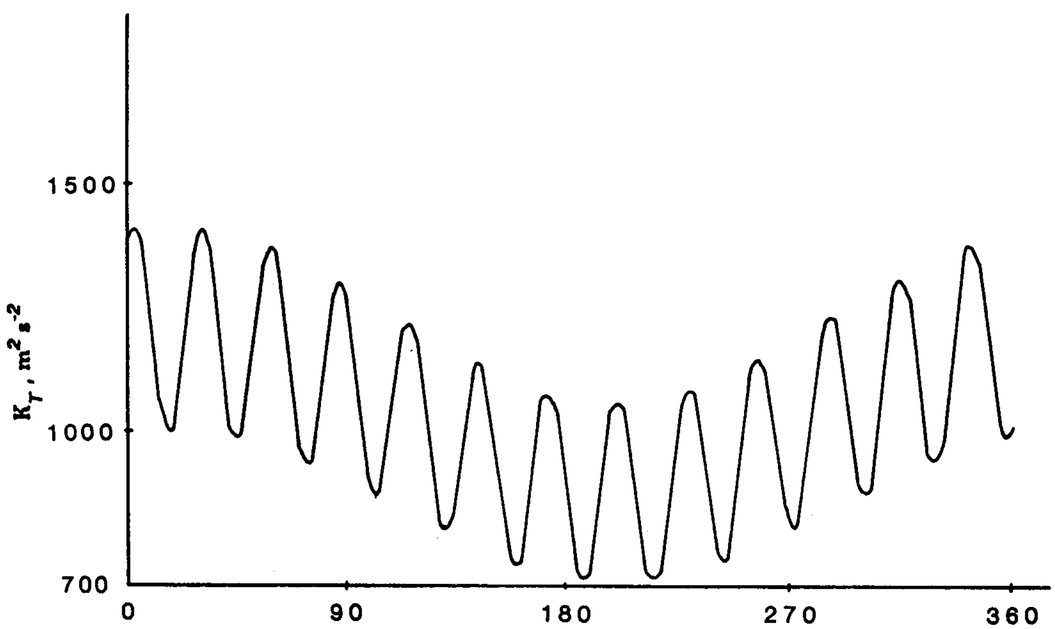

Fig. 5. The kinetic energy of the wave in the vertical shear flow for the eleventh year in the simulation described in Figure 4.

phenomenon. An attempt will be made in the next section to answer the questions raised here by making a closer analysis of the relationships between the two models. We shall, in particular, be interested in how the Lorenz-model can be obtained from the six parameter model.

4. Comparison with the Lorenz model

The purpose of this section is to investigate the connection between the present six parameter model and the simpler three parameter model of Lorenz (1990). For easy reference we start by recalling the equations for the Lorenz model. With a cyclic change of the notations of the variables in such a way that z denotes the zonal variable and x and y the wave variables connected with the cos- and sin-parts of the wave they are:

$$\frac{dz}{d\tau} = -(x^2 + y^2) + a_L(f - z)$$

$$\frac{dx}{d\tau} = zx - b_L zy - x + g$$

$$\frac{dy}{d\tau} = zy + b_L zx - y \tag{4.1}$$

In this model z denotes a nondimensional value of the zonal vertical wind shear or the meridional temperature gradient, while x and y are the amplitudes of the sine and cosine parts of a travelling wave. If the model, described in section 2, shall be reduced to the equations (4.1) we should use the three thermal equations in the system (2.4), originating in the thermal vorticity equation, because these equations contain the description of the external heat sources. The problem is then to relate the variables of the vertical mean flow (i.e. those with subscript $*$) to the corresponding thermal variables. Considering first the zonal equation for the rate of change of B_T it is obvious that one can get the correct form in the first of the Lorenz equations, if one assumes $E_* = F_T$ and $F_* = -E_T$. These assumptions will give a very special structure of the wave, and it is easy to see that the wave will have the temperature field lagging exactly one quarter of a wavelength behind the wave in the streamfunction for the vertical mean flow. Since the heat transport is proportional to the amplitudes of the two waves and to sine of the phase difference, it is seen that the assumption also implies an unchanging phase difference, which at any time for given amplitudes provide a maximum heat transport. The three component model designed by Saltzman *et al.* (1989) is more general than the Lorenz-model, but the same assumption is made with respect to the heat transport and the structure of the wave.

We note further that to obtain the Lorenz form of the first equation we should express the dissipation entirely in the variable B_T. This can be done in a number of ways. One of the reasons for a certain arbitrariness at his point is that the dissipation can be expressed in various ways in a two-level quasi-geostrophic model. It is always required to express the surface vorticity ς_4 in terms of the two vorticities ς_* and ς_T. Phillips (1956) did this by assuming that

$$\varsigma_4 = \varsigma_* - 2\varsigma_T \tag{4.2}$$

which assumes a linear extrapolation. One could also select to do as Charney (1959) who assumed that

$$\varsigma_4 = \varsigma_3/2 = (\varsigma_* - \varsigma_T)/2 \tag{4.3}$$

which is the assumption made in this study. Formally, the two assumptions make a difference in the numerical value of the dissipation coefficient, which in the former case becomes proportional to $\epsilon/4+\epsilon_T$, while the latter case gives $\epsilon/2+\epsilon_T$. However, when the equations are nondimensionalized, it turns out to be a difference in the time used for the scaling of the variables. In any case, we have used a coefficient consistent with the present formulation. Note in this connection that the term containing B_* are disregarded in the dissipation terms.

The two equations for the rate of change of E_T and F_T are treated in a similar way in the dissipation terms, while we use $B_* = B_T$ in the advection terms. Finally we note that the Lorenz model does not contain any beta terms. After all these considerations we may write the equations for the Lorenz model in the form:

$$\frac{dB_T}{dt} = -a_o(E_T^2 + F_T^2) - (e + e_T)B_T + g_z Q_z$$

$$\frac{dE_T}{dt} = a_T B_T F_T - c_T B_T E_T - (\epsilon/4 + \epsilon_T)\frac{b_T}{b_*}E_T + g_e Q_s$$

$$\frac{dF_T}{dt} = -a_T B_T E_T - c_T B_T F_T - (\epsilon/4 + \epsilon_T)\frac{b_T}{b_*}F_T + g_e Q_c \qquad (4.4)$$

(4.4) are the equations as they come from a simplification of the model used in this study, but it has not been possible to deduce the exact form from the two papers by Lorenz (1984, 1990). Indeed, Lorenz states that the equations originally were derived in an *ad hoc manner.* However, once a decision has been made on the assumptions in the two-level quasi-geostrophic model, it is possible to get the Lorenz equations in a form equivalent to (4.4). The nondimensional equations can be obtained by introducing a time scale T, two velocity scales V_z and V_E, and two heating scales S_z and S_E, where the subcripts z and E refer to the zonal and eddy quantities, respectively.

We introduce the following notations:

$$e_1 = \frac{\epsilon b_T}{4b_*} \qquad e_2 = \frac{\epsilon_T b_T}{b_*}$$

$$e = \frac{\epsilon}{4N_o} \qquad e_T = \frac{\epsilon_T}{N_o}$$

$$N_o = 1 + \frac{q^2}{4\lambda^2} \qquad (4.5)$$

Following the normal procedures it is then straightforward to find that the equations in (4.1) are obtained with the following values of the scaling parameters:

$$T = \frac{1}{e_1 + e_2}$$

$$V_z = \frac{e_1 + e_2}{-c_T}$$

$$V_E^2 = \frac{e_1 + e_2}{(-a_o c_T)}$$

$$S_z = \frac{(e + e_T)(e_1 + e_2)}{(-g_z c_T)}$$

$$S_E = \frac{(e_1 + e_2)^2}{(-a_o c_T)_e^g} \tag{4.6}$$

The two constants in the Lorenz equations become:

$$a_L = \frac{e + e_T}{e_1 + e_2}$$

$$b_L = \frac{a_T}{-c_T} \tag{4.7}$$

All the values entering the scaling parameters and the two Lorenz constants are wavelength dependent. To obtain a reasonable agreement with the values used by Lorenz it is required to select the proper value of the wavelength. This choice was made by determining the value of the wave number in such a way that $b_L = 4$ as required. Corresponding to this value we find that $a_L = 0.33$. The value used by Lorenz for this constant is 0.25, but the small discrepancy may be explained in terms of the choice made here for the numerical values of the width of the channel and the Rossby radius of deformation (q). The value of the wavelength turns out to be about 4800 km which falls close to the wavelength of maximum baroclinic instability. We observe also that with this value all the coefficient in the equations are positive except c_T. Due to this fact we find that all the scaling parameters are positive.

The numerical values of the scaling parameters may be computed from the formulas given above. We find: $T = 11.7$ days, $V_z = 7.67ms^{-1}$ $V_E = 10.88ms^{-1}$, $S_z = 3.31Wt^{-1}$ and $S_E = 5.20Wt^{-1}$. Among these values the time scale deviates considerable from the 3 days given by Lorenz. The difference of almost a factor of 4 is not easily explained. The first explanation, which comes to mind would be that Lorenz has used dissipation rates four times larger than those employed here. However, if that were the case, the expressions for the scaling parameters show that the scaling parameters for the winds and the heating rates would be four and sixteen times larger, respectively, and that would bring the values obtained in the time integration of the nondimensional equations out of the atmospheric range. Since the time scale in the Lorenz model as shown is also the dissipation time, it would appear that the value of 3 days is somewhat too small, since the dissipation time normally is assumed to be about 10 days, which is in better agreement with the result obtained here. The other scaling values give reasonable dimensional values for the various parameters. The zonal winds for example are in the Lorenz integrations of a maximum value of about 2 corresponding to roughly $15ms^{-1}$, while the heating rate for the winter ($F = 8$) becomes $26.5Wt^{-1}$ which is in reasonable agreement with those computed from observations as remarked earlier.

It may be of interest to make a more detailed analysis of the Lorenz model as given by the system (4.1). A part of such an analysis has of course already been presented in Lorenz (1984, 1990), but we shall go somewhat further. By setting the three time derivatives in (4.1) to zero we get the equations for the stationary states. The equation for the values of z, the zonal wind shear, becomes a cubic equation, which may be written in the form:

$$G^2 = a(F - \overline{z})((1 + b^2)\overline{z}^2 - 2\overline{z} + 1) \tag{4.8}$$

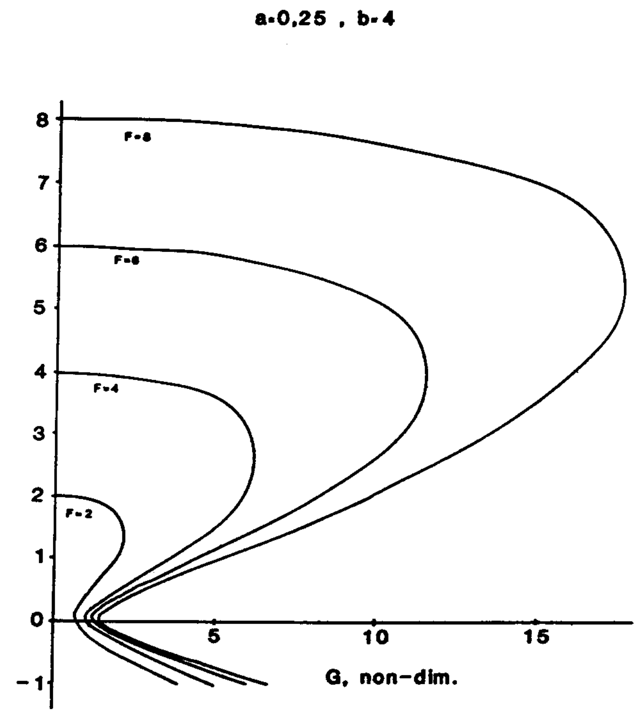

Fig. 6. The steady states for the Lorenz-model. For given values of F and G one obtains the values of z on the ordinate. Note that for large values of F and small values of G only one solution exists.

Since the quadratic expression in the last parenthesis on the right is always positive, it is seen that if $G = 0$ we shall have only one steady state: $\overline{z} = F$. It is thus the inclusion of the heating in the west-east direction, which gives the possibility of having multiple steady states. A graphical representation of the solution to (4.8) are shown in Figure 6, where G has been plotted as a function of $\overline{z}$ for various values of F. For given values of F and G we may use the curves to obtain the steady state values of $\overline{z}$. We note that for large values of F and G we obtain three steady states. However, the time integrations shown by Lorenz (1990) employ the values of $G = 1$, $F = 6$ (summer) and $F = 8$ (winter). Figure 6 shows that for these values we obtain only one steady state with a $\overline{z}$ value very close to, but a little smaller than the given value of F. Denoting

$$D = (1 + b^2)\overline{z}^2 - 2\overline{z} + 1 \tag{4.9}$$

we find the following steady state values for $\overline{x}$ and $\overline{y}$:

$$\overline{x} = \frac{G(1 - \overline{z})}{D} \tag{4.10}$$

$$\overline{y} = \frac{Gb\overline{z}}{D} \tag{4.11}$$

When $\overline{z}$ is close to F we find from (4.10) and (4.11) that the amplitude of the wave in the steady state is very small. For $G = 1$ and $F = 6$ we find an amplitude of 0.04, while $G = 1$ and $F = 8$ give 0.03. As we can see from Figure 6 the steady state values of $\overline{z}$ fall in three classes. One class has negative values of the vertical wind shear. The steady states of this kind are shown in Figure 7 as a function of F and G. This solution does not exist when G is small and F is large, and it is not present in the integrations of the Lorenz equations for values of F larger than 3 when $G = 1$. Figure 7 shows also that the negative vertical wind shear increases as G becomes larger. The two positive solutions for the vertical wind shear may be divided in the small and the large values. The small positive values are shown in Figure 8. This solution is not present in the Lorenz cases since the solution does not exist for $G = 1$ and $F > 3$. Figure 9 shows finally the large positive values of the steady state vertical wind shear. It is seen that the dependence on G is slight, while the value of $\bar{z}$, especially for the larger values of F, is well approximated by F.

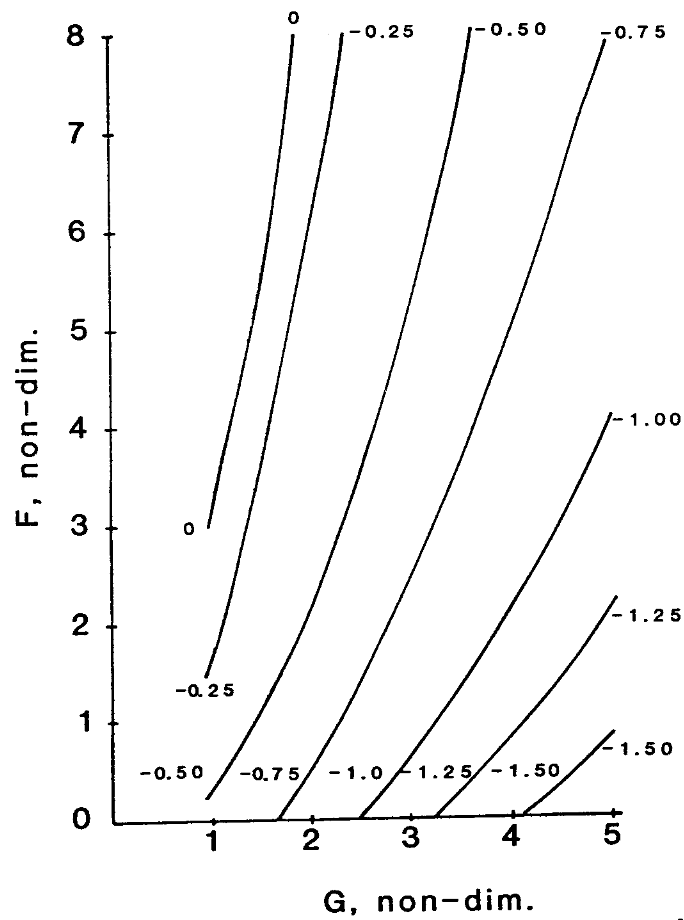

Fig. 7. The steady state solutions with small values of z in the Lorenz-model as a function of F and G. The solution does not exist for large values of F and small values of G, and it indicates an easterly flow for larger values of G.

It is important to know the stability of the classes of steady states which exist for a given pair of G and F. The stability has been determined numerically for all the steady states displayed in Figures 7, 8 and 9 by solving the eigenvalue problem using the method of small perturbations.

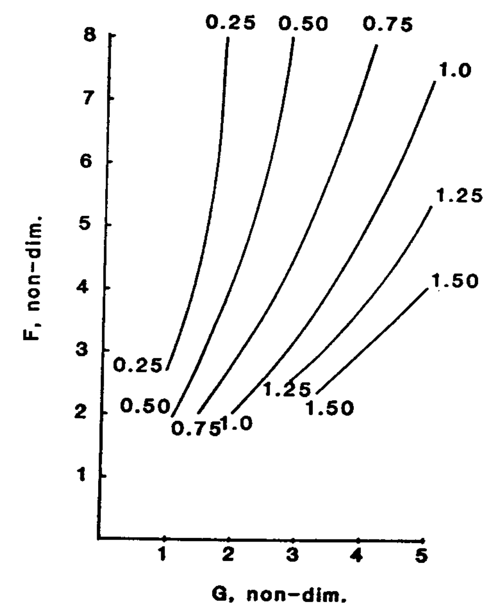

Fig. 8. The steady state solutions with intermediate values of z in the Lorenz-model as a function of F and G. Also this solution is nonexistent for large F and small G. It indicates a westerly flow for large values of G.

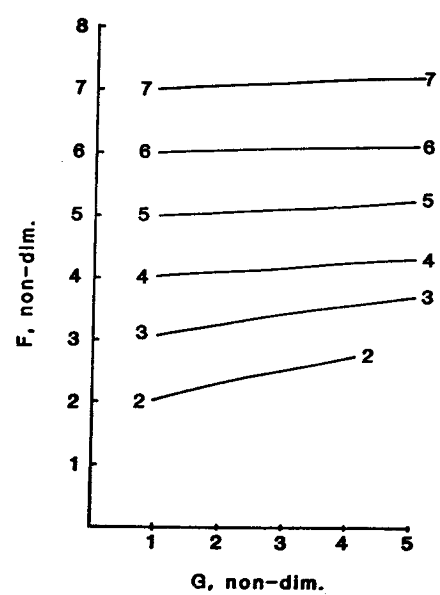

Fig. 9. The steady state solutions with large values of z in the Lorenz-model as a function of F and G. The values for z are only slightly smaller than F.

The results of these calculations are that the steady states having a positive value of the vertical wind shear are unstable whenever they exist. For the negative values of the vertical wind shear in the steady state we find that the steady states are stable except for large values of F combined with relatively small values of G. The latter result is shown in Figure 10 as a function of F and

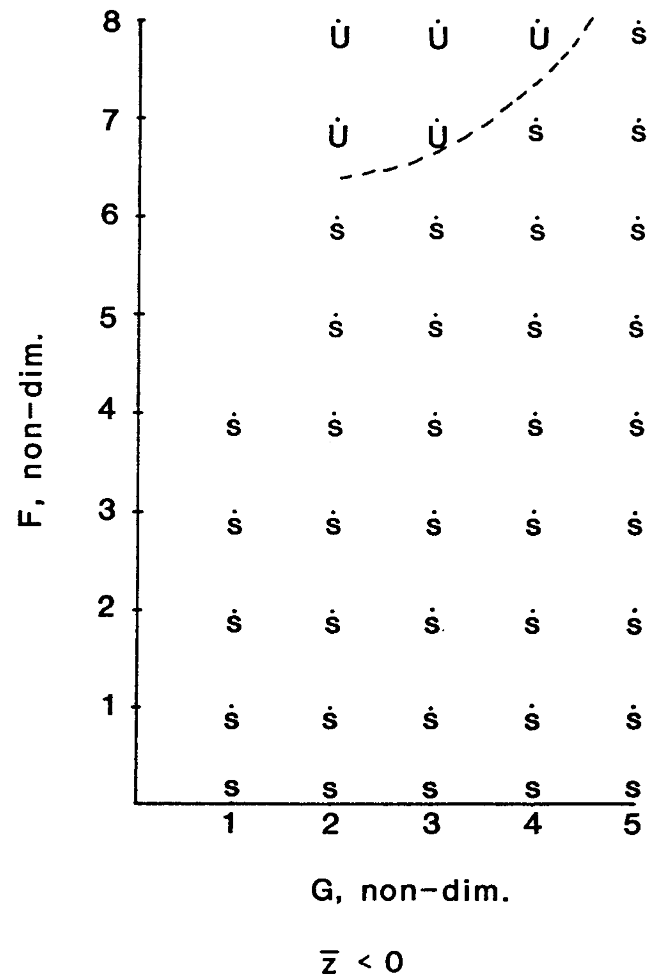

Fig. 10. The stability of the steady state solutions for negative values of z. Instability is obtained only for large values of F and moderate values of G. All other steady states are unstable.

G. Instability is found for $F = 7$ and $F = 8$ for $G = 2$ and $G = 3$, while only $F = 8$ gives instability for $G = 4$. The eigenvalues were determined by a computer program for eigenvalues of non-symmetric matrices, but the eigenvalues are of course also the roots of a cubic equation which for the model is:

$$v^3 + (a + 2(1 - \bar{z})v^2 + (D + 2a(1 - \bar{z}) + 2G^2/D)v$$

$$+\left(aD + \frac{2G^2}{D}(1 - (1 + b^2)\bar{z})\right) = 0 \tag{4.12}$$

To make a partial check on the numerical results one may calculate the conditions under which the constant term in (4.12) is negative, which would indicate that the cubic equation has a positive and real root and thus represent an instability. Leaving out the details we may state the main result that all the smaller positive values of $\bar{z}$ in the steady states belong to this form of solution. With respect to the larger positive values of $\bar{z}$ we have already noticed that they are quite close to the values of F for large values of this quantity. In addition, when G is small, it implies that $\bar{x}$ and $\bar{y}$ are small as well. Under these conditions the frequency equation is reduced to

$$v^3 + (a + 2(1 - F))v^2 + (2a(1 - F) + (1 - F)^2 + b^2F^2)v$$

$$+a((1 - F)^2 + b^2F^2) = 0^2 \qquad (4.13)$$

By direct substitution one may verify that the above equation has the roots

$$v = (F - 1) \pm ibF$$

An excellent agreement is found between the above values and those found from the numerical solutions.

The numerical values of F and G selected by Lorenz (1990) to demonstrate that interannual variations may be due to a chaotic behavior of the atmosphere in the winter season, but a lack of chaos during the summer, are $F = 8$ (winter) and $F = 6$ (summer), while $G = 1$ is used to represent the forcing by heating in the longitudinal direction. We note that with the scaling found earlier in this section $F = 8$ corresponds to $2.65 \times 10^{-2} kJt^{-1}s^{-1} = 26.5Wt^{-1} = 265Wm^{-2}$ where the last number represents the mean heating in an atmospheric column of unit cross section.

F represents the maximum heating in the low latitudes. The heating intensity required to give chaos ($F = 8$) seems rather large compared to the values obtained from observational studies. The latest study (Schaack *et al.*, 1990) based on the ECMWF data from the global weather experiment has not calculated the zonal average of the heating, but looking at the maps it would appear that it cannot be more than 1.5 kJ per day or $174Wm^{-2}$. Other earlier studies (Lawniczak, 1970; Wiin-Nielsen and Brown, 1962) give similar or lower values. For this reason alone it appears doubtful that the atmosphere reaches the chaotic conditions in winter time. It is much more difficult to estimate a proper value of G. Consulting again the maps in the study of Schaack *et al.* (*loc. cit*) it is seen that in addition to the heating contrast between continents and oceans we find also a maximum contrast along the western borders of the Pacific and Atlantic oceans in the regions of the Kurishio and the Gulf currents where the oceans are the heat source in winter time, while the eastern parts of the continents are the sinks. The horizontal dimensions of the latter heating pattern are more appropriate for the horizontal scale selected in the theoretical study which is close to 5000 km. The heating and cooling are especially strong in the Western Pacific and over Eastern China amounting to about 100 Wm^{-2}, which would correspond to a value of G of about 2. The heating and cooling pattern are somewhat less intense in the Western Atlantic and over the eastern part of North America in winter, but would correspond to $G = 1.5$. If these values were representative, Lorenz would have used a too small value of G. The value of G is, however, very important for the behavior of the Lorenz model, because an increase of G by 50 or 100% will change the nature of the solution to a regime in which a stable steady state is present (Fig. 6). Integrations of the Lorenz model equations with $F = 8$ and $G = 2$ show that the trajectory rather soon will approach the stable stationary state and oscillate around it. This state of the atmosphere is rather unrealistic, because it has an easterly zonal current. Figure 11 shows the zonal variable, z, as a function of time in an integration with $F = 8$ and $G = 2$ and with an initial state, which is very close to the unstable steady state with the largest value of $\bar{z}$. We notice that z is very close to the stable steady state in about 24 days.

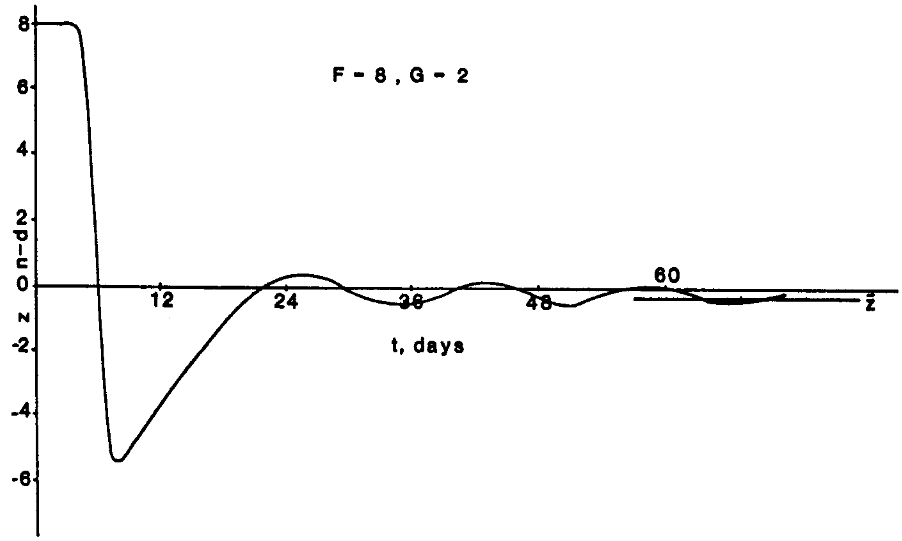

Fig. 11. z as a function of time in a numerical integration for $F = 8$ and $G = 2$ of the equations for the Lorenz-model. The initial state is unstable. The solution approaches the negative, stable, steady state value of z indicated by a horizontal line on the figure.

The arguments given above combined with the fact that the slightly more general model presented in Section 2 has only periodic solutions in those cases where the simpler model gives chaos, indicates that we cannot yet give a definitive answer to the question raised by Lorenz (1990). The additional degrees of freedom present in the model in Section 2, which permits the phase difference between the temperature and geopotential fields to vary freely and the retension of the beta effect, have apparently removed the possibilities for chaotic solutions. To answer the question raised by Lorenz it will probably be necessary to formulate a model with so many degrees of freedom that it will simulate the essential parts of the cascade process in the atmosphere.

5. Concluding remarks

The results of the studies described in this paper show that the particular low-order nonlinear model under investigation has only periodic solutions for the parameter range corresponding to the Earth's atmosphere. The six component, two-level, quasi-geostrophic model has a heating, which depends on latitude and longitude, internal as well as boundary layer friction. At each level one dependent variable is used to describe the zonal flow, while the remaining two variables are the amplitudes of the sine and cosine components of a travelling wave.

The model described above has been compared to the even simpler model used by Lorenz (1990). This three component model is a further simplification of the six-component model, but additional assumptions are naturally necessary to get the simpler model, which describes the development of the thermal part of the slightly more general model. It is thus necessary to relate the dependent variables describing the vertical mean flow to those describing the thermal flow. While the structure of the Lorenz equations are easily obtained, it is more difficult to account for the numerical values quoted by Lorenz, but a reasonable agreement can be obtained except for the scaling of time. The other differences are probably due to the fact that the exact way in which Lorenz has incorporated the friction is unstated in his paper. It is an important aspect of the Lorenz model that the baroclinic wave always has a structure, where the thermal wave is lagging exactly a quarter of a wavelength behind the wave in the geopotential field. For a given amplitude of the thermal wave there is thus a maximum transport of heat by the waves in the Lorenz model. This means that the normal development of a baroclinic wave, where the phase difference between the two fields undergoes systematic changes, is excluded in the Lorenz model due to its simplicity.

Results similar to those described above have been obtained by Saltzman *et al.* (1989). They start from an eight component model, which is very similar to the six component model described in this paper. However, most of their work is done with a three component model, which is a generalization of the model by Lorenz. The main similarity is that the waves are treated in the same way in the two models, and the main difference is the retention of the beta effect in the Saltzman-model.

The tentative conclusion of the investigation by Lorenz is that the chaotic behavior of his model in the winter season combined with the non-chaotic performance during the summer is an internal atmospheric mechanism which may explain the observed interannual variation. It is always difficult to draw conclusions about the real atmosphere from any low-order model. However, the fact that a slightly more general model of the same kind with twice as many dependent variables does not show any sign of chaotic behavior for even large values of the forcing, may indicate that the Lorenz model does not contain a realistic mechanism for the creation of the chaotic states. The six-component model used in this paper does not give an answer to the interesting question raised by Lorenz. The answer may be provided by more general models of the same nature, but containing so many degrees of freedom that the cascade processes in the atmosphere may be simulated in a qualitatively correct manner.

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