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Dynamics of perturbations of the Rossby–Haurwitz wave and the Verkley modon

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RESUMEN

La estabilidad de la onda de Rossby–Haurwitz del subespacio $H_1 \oplus H_n$ y dos tipos de modones de Verkley se analizan a través de la ecuación de vorticidad para un fluido ideal incompresible en una esfera en rotación. Aquí H_n es el subespacio característico del operador de Laplace sobre una esfera, correspondiente al valor característico $n(n+1)$. Se demuestra que perturbaciones arbitrarias de la onda Rossby-Haurwitz se pueden dividir en tres conjuntos invariantes, uno de los cuales contiene un subconjunto estable invariante H_n, tres conjuntos invariantes de pequeñas perturbaciones del modon estacionario de Verkley también son encontrados. La separación de las perturbaciones se logra con la ayuda de una ley de conservación para las perturbaciones.

Fórmulas para la determinación de la distancia entre cualquier pareja de soluciones a partir del conjunto total de modones o de ondas de Rossby-Haurwits se derivan a partir de la energía y enstropia de las correspondientes perturbaciones. Se obtienen condiciones necesarias y suficientes para que la distancia entre estas soluciones sea constante. Se demuestra que cualquier flujo de super–rotación sobre la esfera (perteneciendo a H_1) es estable, independientemente del eje de rotación escogido. Se prueba la inestabilidad de Liapunov para cualquier onda no zonal de Rossby-Haurwitz a partir de $H_1 \oplus H_n$ donde $n \geq 2$ así como para cualquier modon dipolar sobre la esfera. Se demuestra que la inestabilidad de Liapunov es causada por el crecimiento algebraico de la perturbación debido a las oscilaciones asíncronas de las ondas y no tiene que ver con la inestabilidad orbital. Se prueba que cualquier modon monopolar Verkley (1987) con solución exterior que decae rápidamente, así como cualquier polinomio de Legendre son linearmente estables de acuerdo a Liapunov con respecto a subconjuntos invariantes de perturbaciones de una escala suficientemente pequeña.

ABSTRACT

Stability of the Rossby-Haurwitz wave of subspace $H_1 \oplus H_n$ and two types of Verkley's modons is analyzed within the vorticity equation of an ideal incompressible fluid on a rotating sphere. Here H_n is the eigen subspace of the Laplace operator on a sphere corresponding to the eigenvalue $n(n+1)$. It is shown that arbitrary perturbations of the Rossby-Haurwitz wave can be divided into three invariant sets one of which contains a stable invariant subset H_n. Three invariant sets of small perturbations of the stationary Verkley modon are also found. The separation of perturbations have been performed with the help of a conservation law for perturbations.

Formulas for determining the distance between any two solutions from the whole set of modons and Rossby-Haurwitz waves are derived through the energy and enstrophy of the corresponding perturbation. Necessary and sufficient conditions for the distance between these solutions to be constant are obtained. It is shown that any super–rotation flow on a sphere (belonging to H_1) is stable independently of choice of the rotation axis. Liapunov instability of any non–zonal Rossby–Haurwitz wave from $H_1 \oplus H_n$ where $n \geq 2$ as well as of any dipole modon on a sphere is proved. It is shown that the Liapunov instability is caused by the algebraic growth of perturbations due to asynchronous oscillations of waves and has nothing in common with the orbital instability. It is proved that any monopole Verkley (1987) modon, as well as any Legendre polynomial, is linearly Liapunov stable with respect to invariant subsets of perturbations of sufficiently small scale.

1. Introduction

As is known, the large-scale dynamics of the atmosphere can approximately be described by the barotropic vorticity equation. A method for obtaining a series solution to this equation was given by Adem (1956). At present, apart from the stationary zonal flows, there are known two other classes of periodic solutions of this equation for an incompressible ideal fluid on a rotating sphere, namely: the Thompson (1982) waves which include all the Rossby–Haurwitz waves (henceforth, R–H waves), and the solitary waves, or the modons, by Verkley (1984, 1987, 1990) and Tribbia (1984) (Section 2). The stability study of these exact solutions is of considerable interest for deeper understanding the low-frequency variability of the atmosphere.

Many works have been devoted to the barotropic linear instability study of the Rossby wave on the β–plane (Lorenz, 1972; Gill, 1974; Anderson, 1992) and the R–H waves on a sphere (Hoskins, 1973; Baines, 1976; Haarsma and Opsteegh, 1988). In the first part of the present work we summarize the results obtained by Skiba (1989, 1991, 1992) on nonlinear dynamics of arbitrary perturbations of the R–H wave. We emphasize that any perturbation is always the difference of certain two solutions of the vorticity equation, and therefore the differential properties of perturbations are completely determined by those of the solutions. Also, since the system examined is conservative and hence structurally unstable, only exact analytical methods should be used in the stability study to obtain correct results.

Using the conservation law obtained in Section 3, all the possible perturbations of the R–H wave have been divided into three invariant, i.e., independent from each other, sets. Thus it is possible to carry out the stability study separately in each such a set. Both the kinetic energy and the enstrophy of any R–H wave perturbation increase, conserve or decrease with time simultaneously (Proposition 1). Also, it is proved that each non-zonal R–H wave containing the spherical harmonics $Y_n^m(\lambda,\ \mu)$ of degree $n \geq 2$ and $\mid m \mid \geq 1$, is Liapunov unstable (Proposition 11).

The Proposition 11 does not contradict to the results obtained by Hoskins (1973) at the initial stage of interaction of the severely truncated system containing only three waves. Indeed, the Hoskins necessary and sufficient conditions for the linear (exponential) instability of a Rossby wave on the β–plane states that its amplitude has to exceed the critical value. The same statement is also true for the sphere as a sufficient condition. Unfortunately, it is not a necessary condition for the linear instability of the R–H wave because Hoskins used the periodical functions in both x and y directions to obtain the necessary condition on the β–plane, and in case of a sphere this method can not be applied. Therefore, if the R–H wave amplitude is less than the critical value then the stability of the R–H wave is still a problem to be determined. The new result proved here is that under the conditions of Proposition 11 any non-zonal R–H wave is Liapunov unstable independently of its amplitude. It is shown in Section 7 that the instability is caused by the algebraic growth of small initial perturbations. Such perturbations are always existing and caused by the asynchronous oscillations of two R–H waves which slightly differ from each other only by their super-rotational components.

In the second part of our work we examine the stability properties of Verkley's modon. Modon properties are mainly of meteorological interest in connection with the phenomenon of atmospheric blocking (McWilliams, 1980; Haines, 1989; Nezlin and Snezhkin, 1990). The structural stability of the modons has been numerically examined by McWilliams *et al.* (1981), McWilliams and Zabussky (1982), Carnevale *et al.* (1988a, b) and others.

The proof of the linear stability of the Stern (1975) modon by Sakuma and Ghil (1990) was refuted by Ripa (1992) and Nycander (1992). The linear stability of the β–plane modons was studied by Gordin and Petviashvili (1985), Pierini (1985), Laedke and Spatschek (1986, 1988), Swaters (1986), Petviashvili and Pohotelov (1989). But Muzylev and Reznik (1992) and

Nycander (1992) have shown that the proofs by Laedke and Spatschek (1986, 1988), Gordin and Petviashvili (1985) and Petviashvili and Pohotelov (1989) contain serious mistakes. Further, Carnevale *et al.* (1988) refuted the proofs by Pierini (1985) and Swaters (1986). The reasons of the Swaters mistakes are discussed in Section 10. Thus the problem of analytical proof of the linear stability of the two–dimensional modons is so far unsolved.

An extension to the class of barotropic modon solutions on a sphere is offered by Verkley (1987, 1990). Modons previously obtained by Verkley (1984) and Tribbia (1984), represent a modification for a rotating sphere of Larichev and Reznik (1976) modons and can become stationary only in easterly background flows. The new Verkley (1987, 1990) modons, although much less local, can become stationary in westerly background flows that is more realistic for midlatitudes of the atmosphere. Note that these modons have no prototypes on the β–plane because of the different geometrical properties of the β–plane and the sphere. Indeed, the sphere is a closed manifold bounded in the euclidean space R^3, and any two times continuously differentiable streamfunction on the sphere has a finite energy and enstrophy. Unlike it, the β–plane is an unbounded set and hence a modon solution of the vorticity equation must have a rapidly decaying streamfunction $\psi(x,\ y)$ as $\mid x^2 + y^2 \mid \rightarrow \infty$ so as to posses finite values of the energy and enstrophy.

In all numerical experiments carried out by Verkley (1987) using the method of normal modes on a sphere, a stationary modon in a midlatitude westerly zonal flow was linearly unstable. Another Verkley (1990) modon solution has an inner region in which the absolute vorticity is uniform. This solution is similar to atmospheric blockings which are characterized by comparatively low and uniform isentropic potential vorticity in blocking regions. Verkley has proved that the vorticity amplitude of any growing or decaying normal mode is zero within the region of uniform absolute vorticity of the modon. Therefore, this modon, although also linearly unstable (Verkley, 1990), yet compared to the previous Verkley (1984, 1987) modon solutions has the best prospects of survival under the influence of small perturbations.

In order to study the stability of a solution one has to introduce a metric in the phase space for estimating the rate of convergence or divergence of the paths of the solutions. In the present work, as a measure of the distance in phase space between any two solutions $\psi(t)$ and $\tilde{\psi}(t)$ of the vorticity equation we take the square root of a linear combination of the energy and the enstrophy of the streamfunction difference $\psi(t) - \tilde{\psi}(t)$. This choice of the distance is the most natural. Indeed, first the existence and the uniqueness of vorticity equation solutions were proved with these norms (Szeptycki, 1973). And second, according to Fjørtoft (1953), the kinetic energy and the enstrophy are the main invariant functionals of this equation characterizing the nonlinear dynamics of the system. In this connection, if a solution is Liapunov unstable in such a norm, then there is no sense in trying to find some other norm in which the solution would be Liapunov stable.

Analytical formulas have been derived here for the kinetic energy and the enstrophy of the difference between a modon and a R–H wave (Section 5) and between two modons (Section 6). These formulas have been applied to obtain the necessary and sufficient conditions under which the distance between a modon and a R–H wave or between two modons is constant in time. Also in the case with two modons moving along the same latitude circle (Section 6), the formula for the enstrophy has been used in Section 8 to prove the Liapunov instability of any Verkley dipole modon on a sphere (Proposition 12). The proof consists in constructing for the basic modon another modon close to the first one at initial time such that the evolution of the distance between them will contradict the definition of the Liapunov stability. We have only examined Verkley (1984, 1987) modons, however it is easily seen that the same technique can also be used to prove the Liapunov instability of any Verkley (1990) dipole modon. The formulas derived here, also enable us to estimate the growth rate of the energy and the enstrophy of any

unstable perturbation which represents the difference between a modon and a R–H wave (or another modon).

We emphasize that the Liapunov instability of any non–zonal R–H wave or arbitrary dipole modon has nothing in common with the orbital (or Poincaré) instability (for a definition of the orbital instability see, for example, in Zubov (1957)). The main reason of the Liapunov instability is the existence of asynchronous oscillations of waves even if their orbits are very close to each other in the phase space. Actually, because of different phase velocities, the representative points on the orbits of two waves which are very close to each other at initial time, will diverge from each other.

Invariant sets of small perturbations of any stationary Verkley modon are got in Section 9. An invariant functional for small perturbations obtained by Laedke and Spatschek (1986) and Swaters (1986) for the beta–plane modon with rapidly decaying exterior solution, was derived here for more wide class of the modon solution on a sphere containing all Verkley's modons. Note that each modon can be made stationary by choice of corresponding moving coordinate system. Results obtained show that structure of the invariant sets of small perturbations of the R–H wave and the stationary modon are similar to each other. Principal peculiarities of the Liapunov stability analysis in an invariant set of perturbations are shortly discussed in Section 10. In particular, Zubov's criterion of the Liapunov instability in metric space is given here. This criterion is used in Section 11 to show that any Verkley (1987) monopole modon as well as any Legendre polynomial are linearly Liapunov stable with respect to perturbations of sufficiently small scale (Propositions 14 and 15). Linear Liapunov stability of a solution of a nonlinear problem means that this solution is Liapunov stable for the case in which the original problem linearized about such a solution is analyzed.

2. R–H waves and modons on a sphere

The non–dimensional vorticity equation describing the dynamics of an invicid incompressible fluid on a rotating unit sphere S can be written as

$$\frac{\partial}{\partial t}\Delta\psi + J(\psi,\ \Delta\psi + 2\mu) = 0 \tag{1}$$

Here $\psi(t,\ \lambda, \mu)$ is the stream function, $\mu = \sin\varphi$, φ and λ are the latitude and longitude of a point $(\lambda,\ \mu)$ on S, $\Delta\psi$ and $\Delta\psi + 2\mu$ are the relative and absolute vorticity respectively, Δ is the Laplace operator on S and

$$J(\psi,\ h) = \frac{\partial\psi}{\partial\lambda}\frac{\partial h}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial h}{\partial\lambda}$$

is the Jacobian. Let us introduce a Hilbert space $L^2(S)$ of square integrable functions on S with the inner product and the norm defined as

$$< \psi,\ h >= \int_S \psi\bar{h}dS, \qquad \|\psi\| =< \psi, \psi >^{1/2} \tag{2}$$

Here $\bar{h}$ and h are the complex conjugate functions (Marchuk, 1982). It is well–known (Richtmyer, 1982) that

$$L^2(S) = H_o \oplus H_1 \oplus H_2 \oplus \ldots \oplus H_n \oplus \ldots,$$

i.e., $L^2(S)$ is the direct orthogonal sum of the eigen spaces

$$H_n = \{\psi : -\Delta\psi = n(n+1)\psi\} \tag{3}$$

of the Laplace operator Δ. Each H_n corresponds to the eigenvalue

$$\chi_n = n(n+1) \tag{4}$$

and $2n+1$ spherical harmonics

$$Y_n^m(\lambda,\ \mu) = Q_n^m(\mu)e^{im\lambda} \tag{5}$$

of the degree n $(-n \leq m \leq n)$ form an orthonormal basis in H_n:

$$< Y_n^m(\lambda,\ \mu),\ \ Y_n^k(\lambda,\ \mu) >= \delta_{mk}. \tag{6}$$

Here δ_{mk} is the Kroneker symbol,

$$Q_n^m(\mu) = C_{nm}P_n^m(\mu) \tag{7}$$

where

$$P_n^m(\mu) = \frac{(1-\mu^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n \tag{8}$$

is the associated Legendre function of the first kind, and

$$C_{nm} = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}. \tag{9}$$

is the normalizing factor.

Note that every function $f(\lambda,\ \mu)$ of $L^2(S)$ can be represented by its convergent Fourier–Laplace series,

$$f(\lambda,\ \mu) = \sum_{n=0}^{\infty} f_n(\lambda,\ \mu) \equiv \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m Y_n^m(\lambda,\ \mu) \tag{10}$$

where $f_n(\lambda,\ \mu)$ is the orthogonal projection of $f(\lambda,\ \mu)$ on H_n and

$$f_n^m =< f(\lambda,\ \mu),\ \ Y_n^m(\lambda,\ \mu) > \tag{11}$$

is the Fourier coefficient of f. Besides, each eigen subspace H_n is invariant under the Laplace operator and also with respect to any rotation of the sphere (Helgason, 1984). This result, as well as the theoretical results on convergence of Fourier–Laplace series on a sphere (Skiba, 1989) show that mathematically the triangular truncation of a series is more correct operation than the rhomboidal one.

We will make use of the relations

$$< J(\psi,\ f),\ h >=< J(f,\ h),\ \psi >= - < J(\psi,\ h),\ f > \tag{12}$$

$$< J(\psi,\ h),\ F(\psi) >= 0 \tag{13}$$

$$< J(\psi,\ \mu),\ \Delta\psi >=< J(\psi,\ \Delta\psi),\ \mu >= 0 \tag{14}$$

which are valid for arbitrary real sufficiently smooth functions ψ, f and h on a sphere (Skiba, 1989). Here $F(\psi)$ is any continuously differentiable function of ψ (for example, $F(\psi)$ can be a polynomial of ψ). The relation (12) represents the rule of transposition of ψ, f and h. Taking in (13) $F(\psi) \equiv 1$ and $F(\psi) \equiv \psi$ successively, obtain

$$\int_S J(\psi,\ h)dS = 0 \tag{15}$$

and

$$< J(\psi,\ h),\ \psi >= \int_S J(\psi,\ h)\psi dS = 0 \tag{16}$$

Using (12)–(14) it is easy to show that the kinetic energy $\frac{1}{2} \parallel \nabla\psi \parallel^2$, the enstrophy $\frac{1}{2} \parallel \Delta\psi \parallel^2$, the integral vorticity $\int_S \Delta\psi dS$, the angular momentum $< \Delta\psi,\ \mu >\equiv \int_S u\sqrt{1-\mu^2}dS$ and $\int_S F(\Delta\psi + 2\mu)dS$ are constant in time for any solution $\psi(t,\ \lambda,\ \mu)$ of Eq. (1). Here F is an arbitrary function of the absolute vorticity whose differential properties are the same as in the identity (13).

For any constant coefficients ω and f_m the R–H wave

$$f(t,\ \lambda,\ \mu) = -\omega\mu + \sum_{m=-n}^{n} f_m Y_n^m(\lambda - C_n t,\ \mu) \tag{17}$$

consists of the solid body rotation component belonging to H_1 and the homogeneous spherical polynomial of degree n of H_n. The wave (17) of the subspace $H_1 \oplus H_n$ is an exact solution of Eq. (1) if

$$C_n = \omega - \frac{2(\omega+1)}{\chi n} \tag{18}$$

Here ω is the super–rotation velocity and C_n is the speed of the R–H wave (Thompson, 1982). Obviously in each subspace H_n the set of all R–H waves is identical to that of all Thompson waves. Note that

$$\Delta f + 2\mu = -\chi_n f - \chi_n C_n \mu \tag{19}$$

and

$$J(\Delta f,\ f) = (\chi_n C_n + 2)\frac{\partial f}{\partial \lambda} \tag{20}$$

for any R–H wave (17), (18).

In addition to the coordinate system $(\lambda,\ \mu)$, let us consider a primed coordinate system $(\lambda',\ \mu')$ whose pole N' with coordinates $\lambda = \lambda_o$ and $\mu = \mu_o$ moves along the latitude circle with constant velocity C according to the law: $\lambda_o = Ct,\ \mu_o = \text{const}$.

In the primed coordinates the Verkley (1984, 1987) modon has the form

$$\psi(\lambda',\ \mu') = R(\mu')\cos\lambda' + G(\mu') \tag{21}$$

where

$$R(\mu') = A_i P_\alpha^1(\mu') + \omega_i\sqrt{1-\mu_o^2}P_1^1(\mu')$$

$$G(\mu') = B_i P_\alpha^o(\mu') - \omega_i\mu_o P_1^o(\mu') + D_i \tag{22}$$

in the inner region $S_i = \{(\lambda',\ \mu')\epsilon S : \mu' > \mu_a\}$ and

$$R(\mu') = A_o P_\sigma^1(-\mu') + \omega_o\sqrt{1-\mu_o^2}P_1^1(\mu')$$
$$G(\mu') = B_o P_\sigma^o(-\mu') - \omega_o\mu_o P_1^o(\mu') + D_o \tag{23}$$

in the outer region $S_o = \{(\lambda',\ \mu')\epsilon S : \mu' < \mu_a\}$. Here D_i and D_o are constants, the circle $\mu' = \mu_a$ is a boundary between the regions S_i and S_o of the sphere, and the coefficients A_i, A_o and B_i, B_o of the modon are defined as

$$A_i = (C-\omega_o)a_i, \qquad A_o = (C-\omega_o)a_o$$

$$\mu_o A_i = -\sqrt{1-\mu_o^2}B_i, \qquad \mu_o A_o = \sqrt{1-\mu_o^2}B_o \tag{24}$$

where the constants a_i and a_o depend only on the parameters α, σ, μ_a and μ_o:

$$a_i = \frac{b_o(\sigma)}{b_i(\alpha)}\sqrt{1-\mu_a^2}\,\sqrt{1-\mu_o^2}\ \ \frac{1}{P_\alpha^1(\mu_a)}$$

$$a_o = \sqrt{1-\mu_a^2}\,\sqrt{1-\mu_o^2}\ \ \frac{1}{P_\sigma^1(-\mu_a)} \tag{25}$$

We used here the Verkley notation

$$b_i(\alpha) = \chi_\alpha - 2, \quad b_o(\sigma) = \chi_\sigma - 2 \tag{26}$$

Note that the Legendre functions $P_\alpha^m(\mu)$ used here coincide with those of Verkley (1984) except for the factor $(-1)^m$. As a result, the formulas given on the second line of (24) differ by the sign from those obtained by Verkley (1984).

We denote as $\chi_\alpha = \alpha(\alpha+1)$ and $\chi_\sigma = \sigma(\sigma+1)$ the eigenvalues of the Laplace operator corresponding to the spherical harmonics $P_\alpha^m(\mu)\cos m\lambda$ and $P_\sigma^m(\mu)\cos m\lambda$ respectively. The modon (21) is the exact solution of Eq. (1) provided its angular velocity C is

$$C = \omega_o - \frac{2(\omega_o+1)}{\chi_\sigma} \tag{27}$$

(Verkley, 1984). Besides, the dispersion relation

$$-\frac{b_o(\sigma)P_\sigma^1(-\mu_a)}{P_\sigma^2(-\mu_a)} = \frac{b_i(\alpha)P_\alpha^1(\mu_a)}{P_\alpha^2(\mu_a)} \tag{28}$$

between three parameters α, σ and μ_a, is the necessary condition for the modon (21) to exist (Verkley, 1984).

Note that the degree α of the Legendre function $P_\alpha^m(\mu)$ is real ($\alpha \geq 2$) and $\chi_\alpha > 0$. Unlike α, the degree σ may be not only real, but also complex value: $\sigma = -\frac{1}{2} + ik$ where $k > 0$. In the last case the modon is localized only in a small neighbourhood of the inner region S_i, besides, $\chi_\sigma = -(k^2 + \frac{1}{4}) < 0$. In the particular case when $\alpha = \sigma$, the modon (21) is the R–H wave. If $\chi_\alpha \neq \chi_\sigma$ then the Jacobian term of Eq. (1) can be written

$$J(\psi,\ \Delta\psi) = -r(\lambda',\ \mu')\frac{\partial\psi}{\partial\lambda} \tag{29}$$

where

$$r(\lambda',\ \mu') = \begin{cases} C\chi_\alpha + 2, & \text{if } (\lambda',\ \mu')\epsilon S_i \\ C\chi_\sigma + 2, & \text{if } (\lambda',\ \mu')\epsilon S_o \end{cases} \tag{30}$$

Thus the Jacobian is continuous on S only if the modon ψ is stationary (Skiba, 1989). Therefore the stationary modons, as well as the R–H waves, are the classical solutions of the vorticity equation (1), and a non–stationary modon represents the generalized solution of this equation.

If $\mid \mu_o \mid = 1$ then the modon (21) has the only monopole component and respresents a zonal flow in the unprimed coordinate system: $\psi(\mu) = G(\mu)$ (Verkley, 1984). Modon is called dipole if $\mid \mu_o \mid \neq 1$. The dipole modon has the purely dipole structure

$$\psi(\lambda',\ \mu') = R(\mu')\cos\lambda'$$

if $\mu_o = 0$, and both the dipole and monopole components otherwise. At the boundary point $\mu = \mu_a$ the functions $R(\mu)$ and $G(\mu)$ have continuous derivatives only up to the second order. Note also that in case $\sigma = -\frac{1}{2} + ik$, the permissible velocities (18) and (27) of R–H waves and modons do not intersect (Verkley, 1984).

3. Conservation law and invariant sets of perturbations of the R–H wave

In this section, we obtain a conservation law and find invariant, independent, sets of perturbations of the R–H wave (17), (18) (Skiba, 1989). Suppose $\psi(t,\ \lambda,\ \mu)$ is a solution of Eq. (1) different from $f(t,\ \lambda,\ \mu)$. Then their difference

$$\psi'(t,\ \lambda,\ \mu) = \psi(t,\ \lambda,\ \mu) - f(t,\ \lambda,\ \mu) \tag{31}$$

satisfies the equation

$$\frac{\partial}{\partial t}\Delta\psi' + J(f,\ \Delta\psi') + J(\psi',\ \Delta f + 2\mu) + J(\psi',\ \Delta\psi') = 0 \tag{32}$$

with the initial condition

$$\Delta\psi'(0,\ \lambda,\ \mu) = \Delta\psi(0,\ \lambda,\ \mu) - \Delta f(0,\ \lambda,\ \mu) \tag{33}$$

Thus the stability problem of the R–H wave $f(t,\ \lambda,\ \mu)$ is reduced to that of the zero solution of (32).

Let us take the scalar product (2) of Eq. (32) with ψ' and $\Delta\psi'$ successively. Using the relations (12), (14) and (16) we obtain

$$\frac{d}{dt}K(t) = - < J(\psi',\ \Delta\psi'),\ f > \tag{34}$$

$$\frac{d}{dt}\eta(t) =< J(\psi',\ \Delta\psi'),\ \Delta f > \tag{35}$$

where

$$K(t) = \frac{1}{2}\|\nabla\psi'(t)\|^2 \quad \text{and} \quad \eta(t) = \frac{1}{2}\|\Delta\psi'(t)\|^2 \tag{36}$$

are respectively the kinetic energy and the enstrophy of arbitrary perturbation ψ' of the wave f.

Note that time-derivative of the energy of any perturbation which is zonal or belongs to a finite dimensional subspace H_n is zero. The same is true for arbitrary perturbation of the form $F(f)$, i.e., functionally related to the basic R–H wave (17). Here $F(f)$ is a differentiable function of f. Therefore, such perturbations belong to a boundary separating the domains of the generation ($\frac{d}{dt}K(t) > 0$) and dissipation ($\frac{d}{dt}K(t) < 0$) of the perturbation energy in the phase space.

Taking into account (14) and (19), (34) and (35) lead to

$$\frac{d}{dt}\eta(t) = \chi_n\frac{d}{dt}K(t) \tag{37}$$

where χ_n is defined by (4). Thus, we have (Skiba, 1989)

Proposition 1. *Any perturbation of the R–H wave (17), (18) evolves in such a way that its energy $K(t)$ and enstrophy $\eta(t)$ decrease, remain constant or increase simultaneously according to (37).*

As known, the perturbation enstrophy contains information not only about the amplitudes but also about the spectral composition of the perturbation. Therefore, it is generally a stronger integral characteristic than the kinetic energy. However, according to Proposition 1, for analysing the stability of the R–H wave (17), (18) it is sufficient to examine the tendency of the perturbation kinetic energy only without taking into account that of the enstrophy. The functional

$$U[\psi'(t)] = \eta(t) - \chi_n K(t) \tag{38}$$

where $n \geq 1$, is constant due to (37), i.e.,

$$\frac{d}{dt}U[\psi'(t)] \equiv \frac{d}{dt}\{[\chi(\psi') - \chi_n]K(t)\} = 0 \tag{39}$$

for any perturbation ψ' of the wave f. Here χ_n is defined by (4), and

$$\chi(\psi') = \eta(t)/K(t) \tag{40}$$

is the mean spectral number of ψ'. Eq. (39) can be written as

$$\sum_{k=n+1}^{\infty} \chi_k(\chi_k - \chi_n) \sum_{m=-k}^{k} |\psi'^m_k(t)|^2 - \sum_{k=1}^{n-1} \chi_k(\chi_n - \chi_k) \sum_{m=-k}^{k} |\psi'^m_k(t)|^2 =$$

$$= L_o \equiv \sum_{k=n+1}^{\infty} \chi_k(\chi_k - \chi_n) \sum_{m=-k}^{k} |\psi'^m_k(0)|^2 - \sum_{k=1}^{n-1} \chi_k(\chi_n - \chi_k) \sum_{m=-k}^{k} |\psi'^m_k(0)|^2$$

In the space of the perturbation Fourier coefficients $\{(\psi')^m_k\}$ (see (11)) the hypersurface defined by the last equation does not contain coefficients $(\psi')^m_n$ where $|m| \leq n$. It means that the law (39) describes the perturbation dynamics in the subspace orthogonal to H_n.

The property (37) was originally established by Gill (1974) for an infinitesimal disturbance of a stationary planetary Rossby wave on the beta-plane. In case of a sphere, the same property for infinitesimal perturbation of a stationary solution from H_n was obtained by Dymnikov and Filatov (1988). Proposition 1 generalizes these results to arbitrary perturbations of any nonstationary R–H wave of $H_1 \oplus H_n$.

If we introduce the notation

$$\rho(t) = \chi(\psi') - \chi_n \tag{41}$$

then Eq. (39) can be written as

$$\rho(t)K(t) = L_o = \rho(0)K(0) = \ const \tag{42}$$

where the constant L_o is determined by the values of $\chi(\psi')$ and K of an initial perturbation $\psi'(0)$. The law (42) represents a hyperbolic interdependence between the mean spectral number and the kinetic energy of any perturbation of the R–H wave (17), (18).

Due to (42), the whole space of perturbations of the R–H wave (17), (18) can be divided into three invariant sets:

$$M^n_+ = \{\psi' : \chi(\psi') > \chi_n\}$$

$$M^n_o = \{\psi' : \chi(\psi') = \chi_n\}$$

$$M^n_- = \{\psi' : \chi(\psi') < \chi_n\} \tag{43}$$

Remember that, by definition, a set M of perturbations $\psi'(t)$ of a solution is called invariant if $\psi'(t_o) \epsilon M$ implies $\psi'(t) \epsilon M$ for all $t \geq t_o$. Further, according to (37) and (40),

$$\frac{d}{dt}\chi(\psi') = \frac{d}{dt}\{\eta(t)/K(t)\} = \frac{1}{K(t)}\{\chi_n - \chi(\psi')\}\frac{d}{dt}K(t) \tag{44}$$

and hence, the energy cascade of growing perturbations of the R–H wave has the opposite directions in the sets M_-^n and M_+^n: the derivative $\frac{d}{dt}\chi(\psi')$ is positive in M_-^n and negative in M_+^n. Besides, due to (42), the closer the mean spectral number $\chi(\psi')$ to χ_n, the larger is the perturbation energy. Note that, by definition (43), the energy cascade of perturbations in the set M_-^n to the smaller scales is bounded.

Let us now consider all the possible solutions ψ of Eq. (1) which belong to an energy surface with the energy level $K_\psi = \alpha^2 K_f$ where $\alpha > 0$ and K_f is the energy of the basic R–H wave (17). In other words, the solutions considered have the same kinetic energy $\alpha^2 K_f$ defined by the parameter α. Then the kinetic energy of the corresponding set of the perturbations (31) may vary within the limits (Skiba, 1989)

$$K_{min} \equiv (\alpha - 1)^2 K_f \leq K(t) \leq (\alpha + 1)^2 K_f \equiv K_{max} \tag{45}$$

In particular, the minimal level K_{min} and maximal level K_{max} of the perturbation energy correspond to the R–H waves $\psi_A = \alpha f$ and $\psi_B = -\alpha f$ respectively. The case $\alpha = 1$ (when the solutions $f(t)$ and $\psi(t)$ belong to the same kinetic energy surface and the line $K = K_{min} = 0$ coincides with the ρ–axis) is presented in Figure 1, taken from Skiba (1989). During the evolution process a point (ρ, K) showing the current values of the kinetic energy $K(t)$ and the spectral distribution $\chi(\psi')$ of a perturbation (see (41)) may move only along the definite hyperbola (42) in both the directions within the limits (45).

We now prove the next assertion (Skiba, 1989):

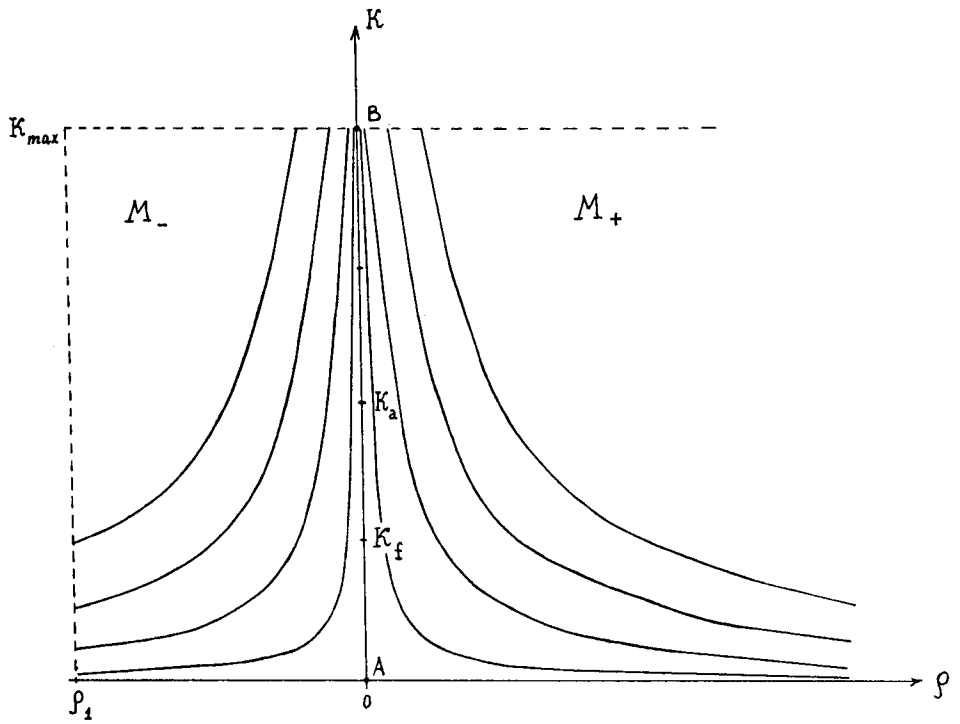

Fig. 1. Relation between the kinetic energy and mean spectral wave number of the R–H wave perturbations ($\alpha = 1$, $K_{min} = 0$).

Proposition 2. *The subspace H_n being a subset of M_o^n, is a stable invariant subset of perturbations of the R–H wave (17), (18).*

Indeed, using (19) we can rewrite Eq. (32) as

$$\frac{\partial}{\partial t}\Delta\psi' + J(f,\ \Delta\psi' + \chi_n\psi') + J(\psi',\ \Delta\psi' - \chi_n C_n\mu) = 0 \tag{46}$$

Therefore, if an initial perturbation ψ' belongs to H_n then it will evolve according to the equation

$$\frac{\partial}{\partial t}\Delta\psi' - \chi_n C_n \frac{\partial}{\partial\lambda}\psi' = 0$$

Obviously, the perturbation will belong to H_n for all time and its energy $K(t)$ and enstrophy $\eta(t)$ will be invariant in time. **QED.**

4. Distance between two R–H waves

By Szeptycki (1973) theorem on the existence and uniqueness of a generalized solution of the vorticity equation, the most suitable norm for determining the distance in the phase space is

$$\|\psi'\|_* = (e_1 K + e_2\eta)^{1/2} \tag{47}$$

where $\psi'(t) = f(t) - \psi(t)$ is the difference between two solutions $\psi(t)$ and $f(t)$ of Eq. (1), the constants e_1 and e_2 are non–negative, and $K(t)$ and $\eta(t)$ are defined by (36). By Proposition 1, we will take $e_2 = 0$ in this section. The energy $K(t)$ of the difference (31) can be written as

$$K(t) = \frac{1}{2}\|\nabla f(t) - \nabla\psi(t)\|^2 = K_a - <\psi(t),\ \Delta f(t)> \tag{48}$$

where

$$K_a = K_\psi + K_f \equiv \frac{1}{2}\|\nabla\psi\|^2 + \frac{1}{2}\|\nabla f\|^2 \tag{49}$$

is the constant as a sum $K_\psi + K_f$ of the kinetic energies of the solutions ψ and f (Fjørtoft, 1953). Due to (48), the energy of the perturbation (31) is completely defined by the current projection of the solution $\psi(t)$ on the R–H wave $f(t)$. Suppose now that $\psi(t)$ is a R–H wave of the subspace $H_1 \oplus H_k$:

$$\psi(t,\ \lambda,\ \mu) = -\tilde{\omega}\mu + \sum_{m=-k}^{k} \psi_m Y_k^m(\lambda - \tilde{C}_k t,\ \mu) \tag{50}$$

Then, by (48),

$$K(t) = K_a - \frac{2}{3}\omega\tilde{\omega} - \delta_{nk}\chi_n \sum_{m=-n}^{m} \psi_m \overline{f}_m e^{im(\tilde{C}_n - C_n)t}\} \tag{51}$$

where δ_{nk} is the Kroneker symbol. The formulas (47) and (51) can be used for determining the distance between any two R–H waves in the phase space. Besides, Eq. (51) leads to

Proposition 3. *Let f be a R–H wave (17) of $H_1 \oplus H_n$ and let ψ be a R–H wave (50) of $H_1 \oplus H_k$ where $n,\ k \geq 2$. Then the energy $K(t)$ of the difference between these two waves is constant in time if and only if at least one of the following conditions is fulfilled:*

1) $k \neq n$;

2) $k = n$ and $\tilde{C}_n = C_n$ (waves oscillate synchronously);

3) $k = n$ and $\psi_m \overline{f}_m = 0$ for all $m \neq 0$.

5. Distance between a R–H wave and a modon

We now derive a formula for determining at any time the distance between a modon (21) and the R–H wave (17) in the phase space of solutions of Eq. (1) (Skiba, 1991). Let us again consider the norm (47) under $e_1 = 1$ and $e_2 = 0$. By (47) and (48) we have

$$\|\psi'(t)\|_*^2 = K(t) = K_a - \chi_n < \psi(\lambda',\ \mu'),\ f(t,\ \lambda,\ \mu) > \tag{52}$$

for the difference $\psi'(t)$ between the modon (21) and the R–H wave (17). Since $< \psi,\ \mu >= \text{const}$, we suppose that $\omega = 0$ (i.e., the R–H wave f belongs to the subspace H_n and $C_n = -\frac{2}{\chi_n} < 0$). The case $\omega \neq 0$ differs from that considered here, by the constant K_a only.

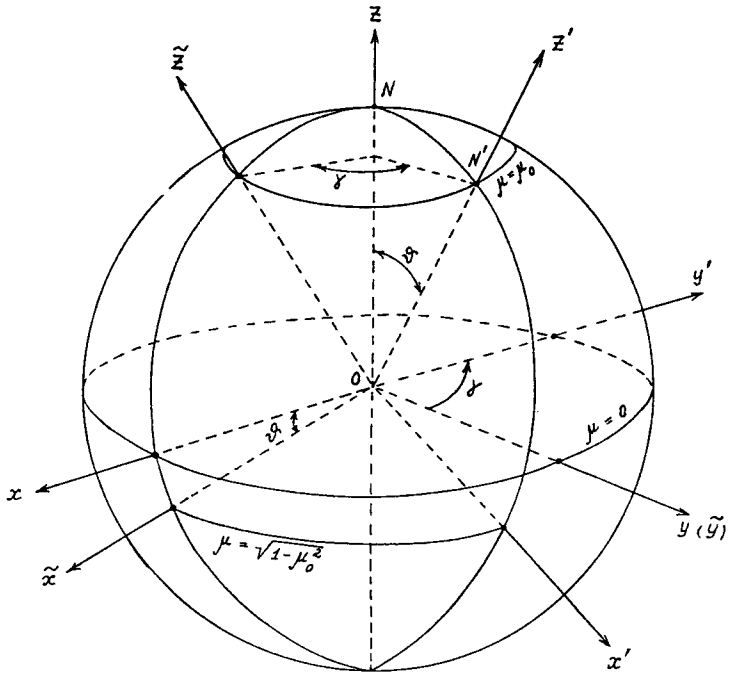

Fig. 2. The fixed coordinate system $(x,\ y,\ z)$ and moving coordinate system $(x',\ y',\ z')$ connected with the Rossby-Haurwitz wave (17) and modon (21) respectively. Coordinate system $(\tilde{x},\ \tilde{y},\ \tilde{z})$ demonstrates a position of system $(x',\ y',\ z')$ at initial time.

Since the modon and the R–H wave are written in the different coordinate systems, we rewrite the R–H wave in the modon coordinates and then calculate the inner product $< \psi(t),\ f(t) >$. To this end, we consider two Cartesian coordinate systems $(x,\ y,\ z)$ and $(x',\ y',\ z')$ whose common origin is in the centre of the sphere S (the point 0 in Fig. 2). Let $(\lambda,\ \mu)$ and $(\lambda',\ \mu')$ be the geographic coordinate systems used for constructing the R–H wave (17) and the modon (21) respectively. Suppose the axes Oz and Oz' of two Cartesian systems coincide with the polar axes ON and ON' of the geographic systems $(\lambda,\ \mu)$ and $(\lambda',\ \mu')$ respectively. Thus the primed system $(x',\ y',\ z')$ rotates together with the modon whereas the system $(x,\ y,\ z)$ is fixed. Denote as $\gamma(t)$ the angle between the axes Oy and Oy'. If originally these axes coincide, i.e., $\gamma(0) = 0$ then

$$\gamma(t) = Ct$$

for any $t \geq 0$ where C is the velocity (27) of the modon. Unlike $\gamma(t)$, the angle ϑ between the axes Oz and Oz' is fixed, besides $\cos\vartheta = \mu_o$.

In order to write the R–H wave (17) in the primed coordinate system $(\lambda',\ \mu')$, we make a rotation of the system $(x,\ y,\ z)$ so as to bring three axes Ox, Oy and Oz into coincidence with the axes Ox', Oy' and Oz' respectively. In our case total rotation consists of two successive elementary rotations of the system $(x,\ y,\ z)$ through the Euler angles γ and ϑ: first about the axis Oz through the angle $\gamma(t)$, and then about the axis Oy' through the angle ϑ (Richtmyer, 1982; Nikiforov and Uvarov, 1984). Note, that after the first rotation the axes Oy and Oy' will coincide with each other.

As a result, in each subspace $H_n (n \geq 1)$, the equations

$$Y_n^m(\lambda,\ \mu) = \sum_{k=-n}^{n} D_{mk}^n(\gamma,\ \vartheta,\ 0) Y_n^k(\lambda',\ \mu') \tag{53}$$

$(|m| \leq n)$ link the spherical harmonics $Y_n^m(\lambda,\ \mu)$ and $Y_n^k(\lambda',\ \mu')$ for two geographic coordinate systems. The generalized spherical functions

$$D_{mk}^n(\gamma,\ \vartheta,\ 0) = \exp\{im\gamma\} d_{mk}^n(\mu_o),$$

also called Wigner's D–functions, form the elements of an unitary matrix $D(\gamma,\ \vartheta,\ 0)$ of finite rotations in the subspace H_n (Nikiforov and Uvarov, 1984). Here

$$d_{mk}^n(\mu) = C_{mk}^n (1-\mu)^{\frac{k-m}{2}} (1+\mu)^{-\frac{k+m}{2}} \frac{d^{n-m}}{d\mu^{n-m}} [(1-\mu)^{n-k} (1+\mu)^{n+k}] \tag{54}$$

and

$$C_{mk}^n = \frac{(-1)^{n-m}}{2^n (n-m)!} \sqrt{\frac{(n+m)!(n-m)!}{(n+k)!(n-k)!}} \tag{55}$$

Taking account of the spatial structure of the modon (21) and the relations

$$\int_o^{2\pi} \cos\lambda \sin m\lambda \ d\lambda = 0 \quad \text{for any} \quad |m| \leq n \tag{56}$$

and

$$\int_o^{2\pi} \cos\lambda \cos m\lambda \, d\lambda = \begin{cases} \pi, & \text{if } m = 1 \\ 0, & \text{if } m \neq 1 \end{cases} \tag{57}$$

we will use the form

$$Y_n^m(\lambda, \mu) = \sum_{k=-1}^{1} D_{mk}^n(\gamma, \vartheta, 0) Q_n^k(\mu') \cos m\lambda' + U_n^m(\lambda', \mu') \tag{58}$$

of Eq. (53) for calculating $< \psi(t), f(t) >$. Here terms $U_n^m(\lambda', \mu')$ are orthogonal to the modon $\psi(\lambda', \mu')$.

Let us use Eq. (21) and the result obtained after substituting Eq. (58) in (17) so as to calculate the inner product $< \psi(t), f(t) >$. Several simple transformations lead to

$$< \psi(t), f(t) >= G_n \sum_{m=-n}^{n} \overline{f}_m d_{mo}^n(\mu_o) \exp\{im(C_n - C)t\}$$

$$+\frac{R_n}{2} \sum_{m=-n}^{n} [d_{m1}^n(\mu_o) - (-1)^{n-m} d_{m1}^n(-\mu_o)] \overline{f}_m \exp\{im(C_n - C)t\} \tag{59}$$

where

$$R_n = \int_{-1}^{1} R(\mu) Q_n^1(\mu) d\mu, \qquad G_n = \int_{-1}^{1} G(\mu) Q_n^0(\mu) d\mu \tag{60}$$

are the Fourier coefficients of the zonal functions $R(\mu)$ and $G(\mu)$ which define the modon (21). To derive (59) we used the relations (Nikiforov and Uvarov, 1984)

$$\begin{aligned} d_{mk}^n(\mu) &= (-1)^{m-k} d_{-m,\,-k}^n(\mu) \\ d_{m,\,-k}^n(\mu) &= (-1)^{n-m} d_{mk}^n(-\mu) \\ d_{mk}^n(\mu) &= (-1)^{m-k} d_{km}^n(\mu) \end{aligned} \tag{61}$$

Fourier coefficients G_n defined by (60) can be excluded from (59) due to the following assertion:

Proposition 4. *Let ψ be any dipole modon (21). Then*

$$G_n = -\chi_n^{-1/2} \frac{\mu_o}{\sqrt{1 - \mu_o^2}} R_n \tag{62}$$

for all Fourier coefficients (60) of ψ.

Proof. Indeed, using the recurrence formulas

$$Q_n^1(\mu) = \chi_n^{-1/2}\sqrt{1-\mu^2}\frac{d}{d\mu}Q_n^0(\mu) \tag{63}$$

and

$$P_\alpha^1(\mu) = \sqrt{1-\mu^2}\frac{d}{d\mu}P_\alpha^0(\mu) \tag{64}$$

for the associated Legendre functions and Eqs. (60) we obtain

$$R_n = -\chi_n^{-1/2}\int_{-1}^{1}\frac{d}{d\mu}\{\sqrt{1-\mu^2}R(\mu)\}Q_n^0(\mu)d\mu \tag{65}$$

and

$$\frac{d}{d\mu}\{\sqrt{1-\mu^2}P_\alpha^1(\mu)\} = \Delta P_\alpha^0(\mu) = -\chi_\alpha P_\alpha^0(\mu) \tag{66}$$

It is easily seen from (22)-(24) and (66) that

$$\frac{\mu_o}{\sqrt{1-\mu_o^2}}\frac{d}{d\mu}\{\sqrt{1-\mu^2}R(\mu)\} = -\Delta G(\mu) \tag{67}$$

Substituting (67) in (65) we have

$$\chi_n^{-1/2}\frac{\mu_o}{\sqrt{1-\mu_o^2}}R_n = \chi_n^{-1}\int_{-1}^{1}\Delta G(\mu)Q_n^0(\mu)d\mu$$

$$= \chi_n^{-1}\int_{-1}^{1}G(\mu)\Delta Q_n^0(\mu)d\mu = -G_n.$$

QED. The next assertion follows immediately from the formula (59) and Proposition 4.

Proposition 5. *The projection* $<\psi(t),\ f(t)>$ *of the R–H wave (17) of* H_n *on the dipole modon (21) is*

$$<\psi(t),\ f(t)> = R_n\sum_{m=-n}^{n}\{\frac{1}{2}[d_{m1}^n(\mu_o) - (-1)^{n-m}d_{m1}^n(-\mu_o)]$$

$$-\chi_n^{-1/2}\frac{\mu_o}{\sqrt{1-\mu_o^2}}d_{mo}^n(\mu_o)\}\overline{f}_m\exp\{im(C_n - C)t\}. \tag{68}$$

In case ψ is a purely dipole modon, $\mu_o = 0$ and the formula (68) is considerably simplified:

$$<\psi(t),\ f(t)> = R_n\sum_{m=-n}^{n}{}' d_{m1}^n(0)\overline{f}_m\exp\{im(C_n - C)t\} \tag{69}$$

The symbol $\sum'$ means that the summation in (69) is performed only over such m that $n-m$ is odd.

Thus Eqs. (52) and (68) (or (69)) enable us to determine at any time t the distance between the dipole modon (21) and the R–H wave (17) from the subspace H_n.

If $\psi(\mu)$ is a monopole modon, then $|\mu_o| = 1$ and $R(\mu) = 0$. Since $d^n_{mo}(\mu_o) = 0$ for all $m \neq 0$ and $d^n_{oo}(\mu_o) = P^o_n(\mu_o)$, the inner product $< \psi,\ f >= G_n f_o P^o_n(\mu_o)$ does not depend on time in this case. That is in agreement with the well–known fact that the norm (47) is constant if ψ' is the difference between a zonal flow and any solution of Eq. (1) which represents a fixed spatial structure moving in the λ–direction with constant velocity (Skiba, 1989). The next assertion is a corollary of Proposition 5 (Skiba, 1991):

Proposition 6. *The enstrophy and the energy of the difference between a R–H wave (17) and a dipole modon (21) are constant if at least one of the following conditions is satisfied:*

a) $R_n = 0$ *(orthogonal solutions);*

b) $f_m = 0$ *if* $m \neq 0$ *(zonal R–H wave);*

c) $C_n = C$ *(synchronous oscillations);*

d) $n = 1$ *(R–H wave belongs to subspace* H_1*);*

e) $\mu_o = 0$ *and simultaneously* $f_m = 0$ *if* $n - m$ *is odd (purely dipole modon and symmetrical (about the equator) R–H wave).*

6. Distance between dipole modons

We now derive a formula for the enstrophy and energy of the difference between two dipole modons (Skiba, 1991). Let the centre of a dipole modon ψ move along the latitude circle $\mu = \mu_o = b$ and the centre of another dipole modon,

$$\tilde{\psi}(\lambda_1,\ \mu_1) = \tilde{R}(\mu_1)\cos\lambda_1 + \tilde{G}(\mu_1), \tag{70}$$

move along the latitude circle $\mu = \mu_o = a$. Let us link the poles N' and N_1 of the geographic coordinate systems $(\lambda',\ \mu')$ and $(\lambda_1,\ \mu_1)$ of the modons ψ and $\tilde{\psi}$ with Cartesian coordinate systems $(x',\ y',\ z')$ and $(x_1,\ y_1,\ z_1)$ (see Fig. 3). Denote as C and $\tilde{C}$ the velocities (27) of ψ and $\tilde{\psi}$ respectively. Suppose the longitudinal angle γ between the poles N' and N_1 at initial time $t_o = 0$ is γ_o. Then it equals

$$\gamma(t) = \gamma_o + (C - \tilde{C})t \tag{71}$$

at any time t. Also denote as β, ρ and ϑ the angles N_1ON', AOB and BOC respectively.

The enstrophy $\eta(t)$ of the difference $\psi'(t) = \psi(t) - \tilde{\psi}(t)$ of two modons is

$$\eta(t) = \frac{1}{2}\|\Delta\psi\|^2 + \frac{1}{2}\|\Delta\tilde{\psi}\|^2 - < \Delta\psi(t),\ \Delta\tilde{\psi}(t) > \tag{72}$$

Here the constant

$$\eta_a = \frac{1}{2}\|\Delta\psi\|^2 + \frac{1}{2}\|\Delta\tilde{\psi}\|^2 \tag{73}$$

is the sum of the enstrophies of two modons.

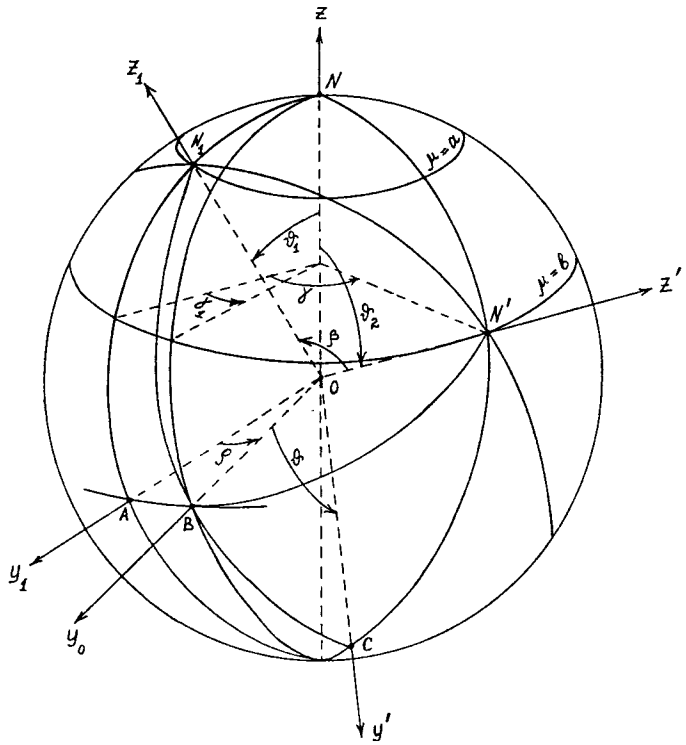

Fig. 3. Disposition of modons (70) and (21) written in geographical coordinate systems $(\lambda_1,\ \mu_1)$ and $(\lambda',\ \mu')$ whose poles N_1 and N' move along the latitude circles $\mu = a$ and $\mu = b$ respectively. The curve AB belongs to the equator $\mu_1 = 0$ and the curve BC is part of the equator $\mu' = 0$. The axis Oy_o is orthogonal to the axes Oz' and Oz_1.

For calculating the inner product in (72) let us rewrite the modon (70) in the system $(\lambda',\ \mu')$ at any time t. To this end, we should make a rotation of the system $(x_1,\ y_1,\ z_1)$ so as to bring it into coincidence with the system $(x',\ y',\ z')$ (see Fig. 3). This rotation can be represented by the matrix $D(\rho,\ -\beta,\ \vartheta)$ (see Nikiforov and Uvarov, 1984) and consists of three successive rotations through the Euler angles $\rho,\ -\beta$ and ϑ which depend on t. These angles are determined in Appendix A.

The rotation $D(\rho,\ -\beta,\ \vartheta)$ results in the relations

$$Y_n^m(\lambda_1,\ \mu_1) = \sum_{k=-n}^{n} D_{mk}^n(\rho,\ -\beta,\ \vartheta) Y_n^k(\lambda',\ \mu') \tag{74}$$

where (Nikiforov and Uvarov, 1984)

$$D_{mk}^n(\rho,\ -\beta,\ \vartheta) = \exp\{i(m+k)\pi\} \exp\{i(m\rho + k\vartheta)\} d_{mk}^n(u) \tag{75}$$

with u defined by (A.1). Since the solution (70) can be written as

$$\tilde{\psi}(\lambda_1,\ \mu_1) = \sum_{n=o}^{\infty} \tilde{R}_n Q_n^1(\mu_1) \cos\lambda_1 + \sum_{n=o}^{\infty} \tilde{G}_n Q_n^0(\mu_1), \tag{76}$$

we will need the formulas

$$Q_n^0(\mu_1) = d_{00}^n(u)Q_n^0(\mu') + 2\cos\vartheta d_{10}^n(u)Q_n^1(\mu')\cos\lambda'$$

$$+U_n^0(\lambda',\ \mu') \tag{77}$$

and

$$Q_n^1(\mu_1)\cos\lambda_1 = -d_{10}^n(u)\cos\rho Q_n^0(\mu')$$

$$+h_n(u)Q_n^1(\mu')\cos\lambda' + U_n^1(\lambda',\ \mu') \tag{78}$$

obtained from Eqs. (74) and (75). Here $U_n^m(\lambda',\ \mu')$ (where $m = 0,\ 1$) are the functions orthogonal to the spherical harmonics $Q_n^0(\mu')$ and $Q_n^1(\mu')\cos\lambda'$ with $n \geq 1$. Besides, $d_{00}^n(u) = P_n^0(u)$, $d_{10}^n(u) = \chi_n^{-1/2}P_n^1(u)$ and

$$h_n(u) = \cos(\rho+\vartheta)d_{11}^n(u) + (-1)^n\cos(\rho-\vartheta)d_{11}^n(-u) \tag{79}$$

Writting both modons in the form (76), using (77) and (78) and taking into account the relations (62) for G_n and $\tilde{G}_n$, we obtain

$$< \Delta\psi(t),\ \Delta\tilde{\psi}(t) >= \sum_{n=1}^{\infty} w_n(u,\ a,\ b)R_n\tilde{R}_n \tag{80}$$

where

$$w_n(u,\ a,\ b) = \chi_n\{\frac{\chi_n}{2}h_n(u) + (\cos\vartheta - \cos\rho)P_n^1(u)$$

$$+\frac{ab}{\sqrt{(1-a^2)(1-b^2)}}P_n^o(u)\} \tag{81}$$

Substituting (80) in (72) we arrive at the following assertion:

Proposition 7. *At each moment t the enstrophy of the difference between the dipole modons (21) and (70) is*

$$\eta(t) = \frac{1}{2}\sum_{n=1}^{\infty}[w_n(1,\ a,\ a)\tilde{R}_n^2 + w_n(1,\ b,\ b)R_n^2$$

$$-2w_n(u,\ a,\ b)R_n\tilde{R}_n] \tag{82}$$

It can similarly be shown that the kinetic energy between two dipole modons is

$$K(t) = \frac{1}{2}\sum_{n=1}^{\infty}\chi_n^{-1}[w_n(1,\ a,\ a)\tilde{R}_n^2 + w_n(1,\ b,\ b)R_n^2$$

$$-2w_n(u,\ a,\ b)R_n\tilde{R}_n] \tag{83}$$

Due to (72), $\eta(t) = \text{Const}$ if and only if the inner product (80) is invariant. Therefore, (82) and (83) lead to

Proposition 8. *The enstrophy $\eta(t)$ and the energy $K(t)$, and hence, the norm (47) of the difference between the dipole modons (21) and (70) are conserved in time if and only if at least one of the following conditions is satisfied:*

a) $\tilde{C} = C$ (synchronous oscillations);

b) $R_n \tilde{R}_n = 0$ for all n (the case of orthogonal modons).

We now consider the particular case when two dipole modons move along the same latitudinal circle $\mu_o = a = b$ (see Fig. 4). Then, according to (A.1)

$$u = \cos\beta = a^2 + (1 - a^2)\cos\gamma(t) \tag{84}$$

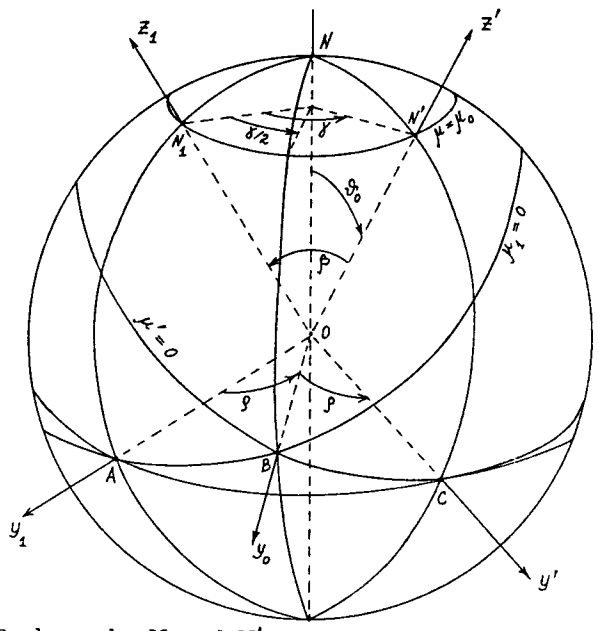

Fig. 4. Special case of Fig. 3 when poles N_1 and N' move along the same latitude circle ($\mu_o = a = b$). Spherical triangles ANB and BNC are equal to each other. Curves AB and BC belong to the equators $\mu_1 = 0$ and $\mu' = 0$ of the systems (λ_1, μ_1) and (λ', μ') respectively.

and $\rho = \vartheta = \gamma/2$. Besides, if follows from (81) and (79) that

$$w_n(u) = w_n(u, a, a) = \chi_n\{\frac{\chi_n}{2} h_n(u) + \frac{a^2}{1 - a^2} P_n^o(u)\} \tag{85}$$

and

$$h_n(u) = \frac{1}{2}\{\cos 2\rho(1 + u) P_{n-1}^{(0,2)}(u) - (1 - u) P_{n-1}^{(2,0)}(u)\} \tag{86}$$

where $P_n^o(u)$ is the Legendre polynomial (8), and

$$P_n^{(\alpha,\,\beta)}(u) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} F(-n;\ n+\alpha+\beta+1;\ \alpha+1;\ \frac{1-u}{2}) \tag{87}$$

is the Jacobi polynomial (see, for example, Nikiforov and Uvarov, 1984) defined through the hypergeometric function

$$F(p; r; s; v) = \sum_{n=o}^{\infty} \frac{(p)_n (r)_n}{(s)_n} \frac{v^n}{n!}, \quad |v| < 1 \tag{88}$$

here $\Gamma(s)$ is the gamma function and $(s)_n = \Gamma(s+n)/\Gamma(s)$ is the Pochhammer symbol.

If poles N_1 and N' of two modons coincide then $\gamma = \rho = \vartheta = 0$, $u = 1$, and hence, $P_n^{(0,\,2)}(1) = 1$ and $h_n(1) = 1$. And if $\gamma = \pi/2$ then $u = a^2$ and $\cos 2\rho = \frac{1-u}{1+u}$ due to (A.6), (A.7). In Section 8, in order to prove the Liapunov instability of the dipole modon we will need the following assertion:

Proposition 9. *Let* $u = a^2$. *Then* $|h_n(a^2)| < 1$. (89)

In combination with (85) the strict inequality (89) will enable us to obtain the inequality (106) and hence select the positive number ε (see (104)). The proof of Proposition 9 is given in Appendix B.

7. Liapunov instability of non–zonal R–H wave

It is well known that the zero solution of (1) is stable in the norm (47) with respect to arbitrary perturbations because of the existence of two conservation laws for the kinetic energy and the enstrophy (Fjørtoft, 1953) The same is also valid for arbitrary solution of the subspace H_1.

Proposition 10. *The kinetic energy and the enstrophy of arbitrary perturbation of any solution of Eq. (1) from the subspace* H_1 *are conserved in time.*

The proof of this assertion is given in Appendix C. Proposition 10 shows that any super-rotational flow on a sphere is Liapunov stable with respect to arbitrary perturbation. But the stability properties of the R–H wave (17) are quite different if $n \geq 2$ (Skiba, 1991). First we review the concept of Liapunov stability of the solution (Liapunov, 1966; Zubov, 1957).

Definition. *A solution* $f(t)$ *of Eq. (1) is Liapunov stable if for any* $\varepsilon > 0$ *and any initial time* t_o *there exists such a number* $\delta = \delta(\varepsilon,\ t_o) > 0$ *and such time* $t_1 \geq t_o$ *that*

$$\|f(t) - \psi(t)\|_* < \varepsilon \tag{90}$$

for all $t \geq t_1$ *and for any solution* $\psi(t)$ *of the equation (1) which satisfies at* $t = t_o$ *the inequality*

$$\|f(t_o) - \psi(t_o)\|_* < \delta \tag{91}$$

The following assertion is proved in Appendix D.

Proposition 11. *If* $n > 1$ *then any non–zonal R–H wave (17) from the subspace* $H_1 \oplus H_n$ *is Liapunov unstable in the norm (47).*

As is known, there are a few mathematical definitions of the solution stability with respect to initial perturbations. In practice, especially in numerical experiments, a concept of the linear (or spectral) stability of a solution is the most popular because of its comparative simplicity (Lorenz, 1972; Gill, 1974, Baines, 1976 and others). In case of a stationary basic solution, the linear stability analysis is carried out by means of the method of normal modes. To this end, an eigenvalue problem is solved for the linearized operator of the problem in order to find the time–space structure of stable and unstable modes in the field of perturbations. There are two main restrictions for the linear stability study, namely: all the amplitudes of perturbations are infinitesimal and only the initial stage of the instability process can be analyzed. If the basic solution is non–stationary then the linear stability analysis becomes more complicated.

From mathematical point of view, the stability of a solution in the Liapunov sense is the most strict and strong concept (Liapunov, 1966). The Liapunov stability guarantees the absence of the exponential as well as algebraic growth of perturbations. Thus a solution can be Liapunov unstable, even if there is no exponential growth of its perturbations. The Liapunov method of the stability study is equally applicable both to non–stationary and stationary basic solutions. Besides, the Liapunov stability characterizes the behavior of perturbations over the whole infinite time interval $(t_o,\ \infty)$ where t_o is an initial time moment. Note also that the Liapunov stability study of any non–stationary R–H wave or a modon, without loss of generality, can be reduced to the stability study of the corresponding stationary solution (see (34), (35) and Proposition 10).

Let us compare the linear stability with the Liapunov stability in the initial stage. The main part of the method of normal modes, the eigenvalue problem, is usually solved numerically or by another approximate method resulting in approximation errors, rounding errors and others. Moreover, a set of matrices whose eigenvectors are linearly dependent, have the zero measure in the whole space of matrices of the same size, i.e., the probability of meeting such a matrix is zero. Hence, because of different errors mentioned above, the matrix of the eigenvalue problem in the method of normal modes is always simple in structure, i.e., its eigenvectors are linearly independent. It means that the algebraic growth of perturbations is excluded from consideration, and only the exponential growth of infinitesimal perturbations is taken into account. Therefore, an approximate method of spectral analysis, especially in weakly unstable cases, can not give full information about the dynamics of the perturbations. It will be shown in this section that just such weak instability always takes place for the non–zonal R–H wave.

We now consider a very simple example so as to bring to light the mechanism of Liapunov instability of non–zonal R–H waves and to illustrate more clearly the proof of Proposition 11. Consider the R–H wave

$$f(t,\ \lambda,\ \mu) = FQ_n^m(\mu)\cos m(\lambda - C_n t) \tag{92}$$

where $Q_n^m(\mu)$ is the normalized associated Legendre function (7), and F is the real amplitude $(m \geq 1,\ n \geq 2)$. The kinetic energy of the wave (92) is $K_f = \pi\chi_n F^2$ where $\chi_n = n(n+1)$. Let $\varepsilon^2 = 4K_f$ (see (D.2)), and let $\delta > 0$ be as small as we like. As another solution of the vorticity equation we take

$$\psi(t,\ \lambda,\ \mu) = -\frac{\delta}{2}Y_1^0(\mu) + FQ_n^m(\mu)\cos m(\lambda - \hat{C}_n t) \tag{93}$$

where $\hat{C}_n$ is the same as in (D.3), and $\hat{C}_n - C_n$ is defined by (D.5).

Remember that m and n are fixed. Therefore, $\hat{C}_n - C_n$ is a very small number. The corresponding perturbation ψ' of the wave (92) is

$$\psi'(t,\ \lambda,\ \mu) \equiv \psi(t,\ \lambda,\ \mu) - f(t,\ \lambda,\ \mu) =$$

$$= -\frac{\delta}{2} Y_1^o(\mu) + A(t) Q_n^m(\mu) \sin m(\lambda - \frac{\hat{C}_n + C_n}{2} t) \tag{94}$$

where

$$A(t) = 2F \sin m \frac{\hat{C}_n - C_n}{2} t = 2F \sin m \frac{\delta(1 - 2/\chi_n)}{4} t \tag{95}$$

Then the kinetic energy $K(t)$ of the perturbation (94) is

$$K(t) = \frac{\delta^2}{4} + \pi \chi_n A^2(t) = \frac{\delta^2}{4} + 4K_f \sin^2\{m \frac{\delta(1 - 2/\chi_n)}{4} t\} \tag{96}$$

Hence, if $t_o = 0$ then the amplitude $A(0) = 0$ and $K(0) = \frac{\delta^2}{4}$, and (91) is fulfilled. However, at $t = \tau_j \equiv \frac{2(1+2j)\pi}{m\delta(1-2/\chi_n)}$ where $j = 0,\ 1,\ 2, \ldots$ we have $A(\tau_j) = 2F$ and (90)is invalid:

$$K(\tau_j) = \frac{\delta^2}{4} + 4K_f = \frac{\delta^2}{4} + \varepsilon^2 > \varepsilon^2. \tag{97}$$

Within the interval $(t_o,\ \tau_o)$, the representative point of the perturbation $\psi'(t)$ will move along a definite hyperbola, according to the law (42), from the point $(\rho_1,\ \frac{\delta^2}{4})$ up to the point whose energy $K(\tau_o)$ is just slightly larger than $4K_f$ (Fig. 1). Thus the amplitude $A(t)$ of the perturbation (94) as well as its kinetic energy $K(t)$ increase monotonically within the time interval $(t_o,\ \tau_o)$. Since the perturbation (94) is periodical it is impossible to choose a moment t_1 so as to satisfy the Liapunov stability definition. Therefore, the R–H wave (92) is Liapunov unstable. This example shows that the instability is possible even if two sums in the left side of the formula next to (40) are invariable. It conforms the point by Shepherd (1988) in his dispute with Petroni *et al.* (1989) on the double cascade mechanism.

We now analyze the result obtained. Due to (95), the amplitude $A(t)$ of the perturbation (94) varies in direct proportion to the amplitude F of the basic solution. Besides, the maximum of the amplitude $A(t)$ is twice that of the basic wave (92) and does not depend on δ. Therefore, if the amplitude F of the basic wave (92) is large then the growth of the amplitude $A(t)$ of the perturbation (94) is considerable and evident independently of the very small initial distance δ between two R–H waves (92) and (93). And if F is small then the growth of $A(t)$ is not so visible. In addition, since δ is very small, the growth of the amplitude $A(t)$ is so slow due to (95) (i.e., the time interval $(t_o\ \tau_o)$ is so large) that in practice the R–H wave can be mistakenly considered as stable. However, by the definition of the Liapunov stability, this wave is unstable. It follows from (95) and (96) that the larger the zonal wavenumber m of the basic solution, the faster is the growth of the amplitude $A(t)$ and the energy $K(t)$ of the perturbation, all other things being equal.

Note that the Liapunov instability of the non-zonal R–H wave has nothing in common with the orbital (or Poincaré) instability. Indeed, due to (91), the orbit of the solution (93) will always be in the tube of the radius δ which envelops the orbit of the basic wave (92). It means that for any time t_1 there is a time t_2 such that the norm (47) of the difference $\psi(t_1,\ \lambda,\ \mu) - f(t_2,\ \lambda, \mu)$ is not larger than δ. The reason for the Liapunov instability is the non-zero shift $\hat{C}_n - C_n$ of

the velocities of two R–H waves whose paths are close to each other in phase space, i.e., the asynchronous oscillations of the waves.

Finally, the choice of the Liapunov functional is essential in the Liapunov stability study. In our opinion, due to the existence and the uniqueness theorem for the vorticity equation solution (Szeptycki, 1973), the norm (47) is the most natural and correct. Anyway, the formulas (94) and (95) give conclusive evidence of the growth in amplitude of the perturbation, and hence, of the instability.

8. Liapunov instability of dipole modons

As an example of using the formula (82) we now show that any dipole modon is unstable in the Liapunov sense. Before formulating the main assertion let us obtain several auxiliary relations. Taking account of (24)-(27) and the formula

$$\omega_i - \omega_o = (C - \omega_o)\{1 - \frac{b_o(\sigma)}{b_i(\alpha)}\} \tag{98}$$

(Verkley, 1984), the function $R(\mu)$ of the modon (21) can be written as

$$R(\mu) = (C - \omega_o)T(\mu) + \omega_o\sqrt{1-\mu_o^2}\sqrt{1-\mu^2} \tag{99}$$

where

$$T(\mu) = \begin{cases} a_i P_\alpha^1(\mu) + \{1 - \frac{b_o(\sigma)}{b_i(\alpha)}\}\sqrt{1-\mu_o^2}\sqrt{1-\mu^2}, & \text{if } \mu \geq \mu_a \\ a_o P_\sigma^1(-\mu), & \text{if } \mu \leq \mu_a \end{cases} \tag{100}$$

is infinitely differentiable function within the intervals $[-1,\ \mu_a)$ and $(\mu_a,\ 1]$, but at $\mu = \mu_a$ it has continuous derivatives up to the second order only. Suppose that the dipole modons (21) and (70) move along the same latitude circle $\mu = \mu_o = a$ (see Fig. 4) and have the same parameters $\alpha,\ \sigma,\ \mu_a$ but different velocities ω_o and $\tilde{\omega}_o$. Then because of (27),

$$C - \tilde{C} = (1 - \frac{2}{\chi_\sigma})(\omega_o - \tilde{\omega}_o) \tag{101}$$

for the modons (21) and (70). Further, due to (22)-(26), the functions $R(\mu)$ and $\tilde{R}(\mu)$ of these modons have the form (99) with the same function $T(\mu)$, and hence,

$$R(\mu) - \tilde{R}(\mu) = (\omega_o - \tilde{\omega}_o)F(\mu) \tag{102}$$

where

$$F(\mu) = -\frac{2}{\chi_\sigma}T(\mu) + \sqrt{1-\mu_o^2}\sqrt{1-\mu^2} \tag{103}$$

We are now in a position to prove the following assertion (Skiba, 1991):

Proposition 12. *Any dipole modon (21) is Liapunov unstable in the norm (47).*

Proof. Since the norm generated by the enstrophy $\eta(t)$ is stronger than that generated by the energy $K(t)$, we take the norm (47) with $e_1 = 0$ and $e_2 = 1$. Let $t_o = 0$ and

$$\varepsilon = \{\frac{1}{2}\sum_{n=1}^{\infty}[w_n(1) - |w_n(a^2)|]R_n^2\}^{1/2} \tag{104}$$

where R_n and $w_n(u)$ are defined by (60) and (85) respectively and $a = \mu_o$ is the latitude of the pole N' (Fig. 4). Due to (85)-(88),

$$w_n(1) = \chi_n\{\frac{\chi_n}{2} + \frac{a^2}{1-a^2}\} > 0 \tag{105}$$

Taking into account Eqs. (85), (105), Proposition 9, and the estimate $|P_n^o(u)| \leq 1$, we obtain

$$|w_n(a^2)| < w_n(1) \tag{106}$$

Thus $\varepsilon > 0$. Let δ be an arbitrary positive number, as small as we like.

We now use the same method as in the Proposition 11 proof. Namely, we show that for ε and any δ chosen, there always exists a solution $f(t) = f(t, \delta)$ of Eq. (1) such that the condition (91) is fulfilled but despite that, it is impossible to find a moment t_1 so as to satisfy the inequality (90) for all $t > t_1$. It gives evidence that the dipole modon $\psi(t)$ is Liapunov unstable.

As a solution $f(t)$ we take the modon (70) whose parameters α, σ, μ_a and $\mu_o = a$ coincide with those of the modon (21), and besides,

$$\omega_o - \tilde{\omega}_o = \delta/\{\sum_{n=1}^{\infty} w_n(1)F_n^2\}^{1/2} \tag{107}$$

Since the functions $F(\mu)$ and $T(\mu)$ and all their derivatives up to the second order inclusive are continuous within the interval $[-1, 1]$, the series in the formula (107) converges to a finite value (Skiba, 1989). Moreover, since δ is small, due to (24)-(26) and (101), two modons will have the slightly different speeds C and $\tilde{C}$ and the amplitudes A_o, A_i and $\tilde{A}_o$, $\tilde{A}_i$.

Suppose that the poles N' and N_1 of the modons (21) and (70) coincide originally at $t_o = 0$, *i.e.*, $\gamma(0) = \gamma_o = 0$. Then, according to (82),

$$\|\psi'(0)\|_*^2 = \eta(0) = \frac{1}{2}\sum_{n=1}^{\infty} w_n(1)\{R_n - \tilde{R}_n\}^2 = \frac{1}{2}(\omega_o - \tilde{\omega}_o)^2\sum_{n=1}^{\infty} w_n(1)F_n^2$$

Further, due to (107), $\eta(0) = \frac{\delta^2}{2}$, and hence, the condition (91) is fulfilled. However, if $t = \tau_j$ where

$$\tau_j = \frac{\pi(2j+1)}{2(\omega_o - \tilde{\omega}_o)(1 - \frac{2}{\chi_\sigma})}, \quad (j = 0, 1, 2, \ldots) \tag{108}$$

then $\gamma(\tau_j) = \pi/2$ due to (71) and (101). Therefore, $u = a^2$ (see Section 6 just before Proposition 9), and (82) and (104) lead to

$$\|\psi'(\tau_j)\|_*^2 = \eta(\tau_j) = \frac{1}{2}\sum_{n=1}^{\infty} w_n(1)\{R_n^2 + \tilde{R}_n^2\} - \sum_{n=1}^{\infty} w_n(a^2)R_n\tilde{R}_n$$

$$\geq \frac{1}{2}\sum_{n=1}^{\infty}\{w_n(1) - |w_n(a^2)|\}\{R_n^2 + \tilde{R}_n^2\}$$

$$= \varepsilon^2 + \frac{1}{2}\sum_{n=1}^{\infty}\{w_n(1) - |w_n(a^2)|\}\tilde{R}_n^2 \tag{109}$$

Thus $\eta(\tau_j) > \varepsilon^2$ due to (106), and the inequality (90) is false at $t = \tau_j$. Since the sequence $\{\tau_j\}$ tends to ∞ as $j \to \infty$, it is impossible to find a moment t_1 to satisfy the Liapunov stability definition. **QED.**

9. Invariant sets of perturbations of stationary modon

By definition, a set M of perturbations $\psi'(t)$ of a particular solution is called invariant if the initial condition $\psi'(t_o)\epsilon M$ implies $\psi'(t)\epsilon M$ for all $t \geq t_o$. As shown in Section 3, all kinds of perturbations of any R–H wave can at least be divided into three invariant, independent, sets.

We now find invariant sets of small perturbations of any stationary Verkley's modon. Unlike the case of the R–H wave, the invariant sets will be obtained here only in a small neighbourhood of the Verkley modon. As before, we construct a nonlinear invariant functional which is analogous to that obtained by Laedke and Spatschek (1986) and Swaters (1986) for the beta–plane modon. It is easy to show that the equation

$$\frac{\partial}{\partial t}\Delta\psi' + J(\psi + \psi',\ \Delta\psi') + q(x)\{J(\psi, \psi') - C\frac{\partial}{\partial\lambda}\psi'\} = 0 \tag{110}$$

holds in the domains S_i and S_o for any perturbation ψ' of an arbitrary modon (21). Here C is the volocity (27) and

$$q(x) \equiv q(\lambda',\ \mu') = \begin{cases} \chi_\alpha, & \text{if } (\lambda',\ \mu')\epsilon S_i \\ \chi_\sigma, & \text{if } (\lambda',\ \mu')\epsilon S_o \end{cases} \tag{111}$$

because of (29). It is known (Szeptycki, 1973) that not only the classical but also the generalized solutions of Eq. (1) have continuous derivatives on the sphere up to the second order. Since any perturbation ψ' satisfying the Eq. (110) is the difference between two solutions of Eq. (1), see (31), it is assumed here that all the derivatives of ψ' up to the second order are continuous on the sphere S. Due to (12) and (34), the perturbation energy equation for the modon can be written as

$$\frac{d}{dt}K(t) = - < J(\psi,\ \psi'), \Delta\psi' > \tag{112}$$

Let

$$\eta(t) = \eta_i(t) + \eta_o(t) \tag{113}$$

where

$$\eta_i(t) = \int_{S_i} |\Delta\psi'|^2 dS, \qquad \eta_o(t) = \int_{S_o} |\Delta\psi'|^2 dS \tag{114}$$

are the portions of the perturbation enstrophy corresponding to the inner and outer regions S_i and S_o respectively. Multiplying Eq. (110) by $\Delta\psi'$ and integrating the result first over S_i and then over S_o, we obtain

$$\frac{d}{dt}\eta_i + \chi_\alpha \int_{S_i} \{J(\psi,\ \psi') - C\frac{\partial}{\partial\lambda}\psi'\}\Delta\psi' dS$$

$$= -\int_{S_i} J(\psi + \psi',\ [\Delta\psi']^2) dS \tag{115}$$

and

$$\frac{d}{dt}\eta_o + \chi_\sigma \int_{S_o} \{J(\psi,\ \psi') - C\frac{\partial}{\partial\lambda}\psi'\}\Delta\psi' dS$$

$$= -\int_{S_o} J(\psi + \psi',\ [\Delta\psi']^2) dS \tag{116}$$

Integrating by parts the last term of both Eq. (115) and Eq. (116), and combining the equations obtained with Eq. (112), we have

$$\frac{d}{dt}\{K - \chi_\alpha^{-1}\eta_i - \chi_\sigma^{-1}\eta_o\} = (\chi_\sigma^{-1} - \chi_\alpha^{-1}) \int_o^{2\pi} Z(\lambda',\ \mu_a) d\lambda' \tag{117}$$

where the function

$$Z(\lambda', \mu') \equiv Z(x) = \left[\frac{\partial\psi}{\partial\lambda'}(x) + \frac{\partial\psi'}{\partial\lambda'}(x)\right] [\Delta\psi'(x)]^2 \tag{118}$$

is continuous on S. We emphasize that (117) is true for any perturbation ψ' of an arbitrary modon (21).

Let $\chi_\alpha = \chi_\sigma = \chi_n$, i.e., ψ is a R–H wave. Then (114) and (117) imply (37), and again we make sure that the functional (38) is invariant.

Suppose now that $\chi_\alpha \neq \chi_\sigma$. Using the expressions (21)–(26) and (98), we obtain

$$\frac{\partial\psi}{\partial\lambda'}(\lambda',\ \mu_a) = -C\sqrt{1-\mu_a^2}\ \sqrt{1-\mu_o^2}\sin\lambda' \tag{119}$$

for the modon (21). Thus

$$\frac{\partial\psi}{\partial\lambda'}(\lambda',\ \mu_a) = 0 \tag{120}$$

for any stationary ($C = 0$) or monopole ($1 - \mu_o^2 = 0$), modon ψ, i.e., the boundary $\mu' = \mu_a$ between S_i and S_o is a streamline for such modons. Hereafter we will not distinguish in this section the stationary and monopole modons.

We now assume that perturbation ψ' is small. Neglecting the second term in (118) and taking into account (120), we obtain $Z(\lambda',\ \mu_a) = 0$. Then (117) leads to the conservation law

$$\frac{d}{dt}U[\psi'(t)] \equiv \frac{d}{dt}\{K - \chi_\alpha^{-1}\eta_i - \chi_\sigma^{-1}\eta_o\} = 0 \tag{121}$$

for small perturbations of the stationary modon (21).

Let us define two functions $\chi_i(t)$ and $\chi_o(t)$ through

$$\eta_i(t) \equiv \chi_i(t)K(t), \qquad \eta_o(t) \equiv \chi_o(t)K(t) \tag{122}$$

Then $\chi(t) = \chi_i(t)+\chi_o(t)$ is the perturbation mean spectral number and all small perturbations of the stationary modon are divided into three invariant subsets M_+, M_o and M_- defined as

$$M_- = \{\psi' \ :\ p(\psi') < 1\}$$

$$M_o = \{\psi' \ :\ p(\psi') = 1\}$$

$$M_+ = \{\psi' \ :\ p(\psi') > 1\} \tag{123}$$

where the non-dimensional number

$$p(\psi') = \chi_\alpha^{-1}\chi_i(t) + \chi_\sigma^{-1}\chi_o(t) \tag{124}$$

characterizes the spectral composition of the perturbation ψ'.

Using (112), (115) and (122), we obtain

$$\frac{d}{dt}p(\psi') = \frac{1}{K(t)}\{1 - p(\psi')\}\frac{d}{dt}K(t) \tag{125}$$

If $\chi_\alpha = \chi_\sigma$ then (125) is identical to (44). If $\chi_\alpha \neq \chi_\sigma$ then (125) is valid only for small perturbations. By (125), the energy cascade of growing perturbations of the stationary modon has the opposite directions in the sets M_- and M_+. Since $U[\psi'(t)] = K(t)[1-p(\psi')]$ is constant, the closer $p(\psi')$ to 1, the larger is the perturbation energy. Thus there is a structural similarity between the invariant sets of small perturbations of the stationary modon and the R–H wave.

10. Liapunov stability in an invariant set

Remember that the Liapunov method for the stability study means an analysis of the stability properties of a solution with respect to all the perturbations from a sufficiently small neighbourhood of the solution. Sometimes, it is interesting to consider stability of a solution only with

respect to a certain subset of perturbations of such a neighbourhood (Swaters, 1986). In this connection, we would like to emphasize that the Liapunov stability method can be used correctly to this subset of perturbations only if it is the invariant set. Invariant sets of perturbations of the R–H wave and the stationary modon have been found in the sections 3 and 9.

It is easy to show that the difference of two elements from the invariant set M_+^n defined by (43), does not always belong to M_+^n. The same is also true for the invariant set M_-^n. Hence the invariant sets M_+^n and M_-^n are not linear spaces. Therefore, if we study the Liapunov stability of R–H wave in the set M_+^n (or M_-^n) only, then this set must be considered as the metric space. The Liapunov stability method in the metric space was developed by Zubov (1957).

Since Eq. (110) reduces the stability study of the modon ψ to that of the zero solution, we now give Zubov's criterion for the zero solution to be Liapunov stable in the metric space.

Proposition 13. *The zero solution is Liapunov stable if and only if there exists a differentiable positive functional* $V[\psi'(t)]$ *defined in a small neighbourhood* $O_r(0) = \{\psi' : \rho(\psi', 0) \leq r\}$ *of the zero and such that*

1. $C_1\rho(\psi', 0) \leq V[\psi'(t)] \leq C_2\rho(\psi', 0)$ *for all* $\psi'(t)\epsilon O_r(0)$;
2. $\frac{d}{dt}V[\psi'(t)] \leq 0$ *for all* $t \geq t_o$ *while* $\psi'(t)\epsilon O_r(0)$.

This statement is a particular case of Theorem 12 by Zubov (1957). The metric $\rho(\psi', 0)$ denotes the distance between the zero solution and a perturbation $\psi'(t)$. We emphasize that the constants C_1 and C_2 are independent of time.

Recall that the proof of Proposition 10 was based on using the properties of the orthogonal projector on H_n in the space $L^2(S)$. But this assertion can also be proved by applying Proposition 13. Let $\eta(t)$ and $K(t)$ defined by (36) be the energy and the enstrophy of perturbation ψ' of a solution ψ of H_1. Then the functional $U[\psi'(t)] = \eta(t) - 2K(t)$ is conserved in time for any ψ' due to (39). Further, the projection of $\psi'(t)$ on the subspace $H_o \oplus H_1$ is also constant in time because of (C.2), (C.3) and invariability of $< \psi', \mu >$. Thus we can assume, without loss of generality, that each perturbation ψ' is orthogonal to $H_o \oplus H_1$. Therefore, we can use the inequality

$$\|\nabla g\| \leq 6^{-1/2}\|\Delta g\| \tag{126}$$

that holds for any function $g(\lambda, \mu)$ of $L^2(S)$ that is orthogonal to $H_o \oplus H_1$ (Skiba, 1989). Then we obtain

$$0 < C_1\eta(t) \leq U[\psi'(t)] \leq C_2\eta(t) \tag{127}$$

where $C_1 = \frac{2}{3}$ and $C_2 = 1$. This estimate means that two different metrics introduced in the space of perturbations by means of $U[\psi']$ and η, are equivalent to each other at any time. Since $U[\psi'(t)]$ is constant, ψ is Liapunov stable due to Proposition 13.

We now show that the estimate (127) is false in the set M_-^n if $n \geq 2$. Indeed, the proof of Proposition 11 shows that any nonzonal R–H wave (17) of $H_1 \oplus H_n (n \geq 2)$ is Liapunov unstable with respect to a certain perturbation ψ' from the invariant set M_-^n. At the same time, because of (39) and (43),

$$\frac{d}{dt}\{-U[\psi'(t)\} = 0 \quad \text{and} \quad -U[\psi'(t)] > 0 \tag{128}$$

for any perturbation ψ' of M_{-}^{n}, and hence, the second condition of Proposition 13 is satisfied. Therefore, due to Proposition 13, the estimate (127) is false in the set M_{-}^{n}.

Note that the sets of the modon perturbations considered by Swaters (1986) are non–invariant and they are not linear spaces. Besides, he proved the estimate $C_1K(0) \leq V[\psi'(t)] \leq C_2K(0)$ instead of $C_1K(t) \leq V[\psi'(t)] \leq C_2K(t)$. Therefore, Zubov's criterion (Proposition 13), can not be used in this case.

11. Stable invariant sets of small perturbations of the monopole modon and the Legendre polynomial

It is known that the norm (47) of the difference between a zonal flow on a sphere and a R–H wave (or a modon) is conserved in time. According to Kuo (1973) criterion, all the Legendre polynomial $P_n(\mu)$ of degree $n \geq 3$ satisfy the necessary condition for the linear instability. Numerical analysis of the linear stability of the Legendre polynomials was carried out by Baines (1976).

The linear stability of monopole modons representing zonal flows and being a combination of two Legendre polynomials of real degrees can be analyzed in the same way.

We now extract invariant subsets of stable small perturbations of the Legendre polynomials and monopole modons. We use Proposition 13 in order to show that each monopole modon with $\chi_\sigma > 0$ as well as an arbitrary Legendre polynomial are linearly Liapunov stable (see the definition in Section 1) with respect to any small amplitude perturbation whose spatial scale is small enough.

Let us fix an integer number m and denote as I_m the subspace of all possible functions on the sphere S whose zonal wavenumber is m. Spherical harmonics $\{Y_n^m(\lambda, \mu) : n \geq |m|\}$ form an orthonormal basis in I_m. As is known, each I_m is the invariant set for infinitesimal perturbations of any zonal flow on S. Let us define for any natural k the invariant set

$$F_k = \bigoplus_{|m| \geq k} I_m \tag{129}$$

as the direct orthogonal sum of such subspaces I_m that $|m| \geq k$, i.e., F_k contains only the functions $f(\lambda, \mu)$ of $L^2(S)$ whose Fourier coefficients f_n^m, see (11), are zero if $|m| < k$.

Proposition 14. *Let ψ be a monopole modon with $\chi_\sigma > 0$, and let $a = max\{\chi_\alpha, \chi_\sigma\}$. Then ψ is linearly Liapunov stable with respect to any initial perturbation of F_k if $k(k+1) > a$.*

Proof. Let $\chi_k = k(k+1)$. We have $p(\psi') \geq \chi(\psi')/a \geq \chi_k/a > 1$, and hence, F_k is the invariant subset of M_+. Further, the functional

$$V[\psi'(t)] = -U[\psi'(t)] = K(t)\{p(\psi') - 1\}$$

satisfies the conditions (128), and thus, the second condition of Proposition 13 is fulfilled. We now estimate $V[\psi'(t)]$. Let $b = min\{\chi_\alpha, \chi_\sigma\}$. Then we obtain

$$V[\psi'(t)] \geq K(t)\left\{\frac{\chi(\psi')}{a} - 1\right\} \geq \eta(t)\left\{\frac{1}{a} - \frac{1}{\chi(\psi')}\right\} \geq \eta(t)\left\{\frac{1}{a} - \frac{1}{\chi_k}\right\}$$

and

$$V[\psi'(t)] \leq K(t)\frac{\chi(\psi')}{b} = \frac{1}{b}\eta(t)$$

Thus the first condition of Proposition 13 is also valid, and the monopole modon is stable to small perturbations of F_k. **QED.**

Proposition 15. *The Legendre polynomial $\psi = P_n(\mu)$ is linearly Liapunov stable with respect to any initial perturbation of the invariant set F_k if $k \geq n$.*

Proof. If $k > n$ then this assertion follows from Proposition 14 as the particular case when $\chi_\alpha = \chi_\sigma = \chi_n$. Let $k = n$. Obviously, it is sufficient to prove the stability of $P_n(\mu)$ regarding the small perturbations of the set I_n only. By Proposition 2, the Legendre polynomial $P_n(\mu)$ is Liapunov stable with respect to perturbations of the invariant set H_n. Therefore, $P_n(\mu)$ is linearly stable with respect to perturbations from the set $H_n \cap I_n$. The set $H_n \cap I_n$ is invariant as it is the intersection of invariant sets. It is a one-dimensional set with the single spherical harmonic $Y_n^n(\lambda, \mu)$ as its basis. Thus, without loss of generality, one can consider only the perturbations of I_n which are orthogonal to $Y_n^n(\lambda, \mu)$. Then again $\chi(\psi') > \chi_n$, and the same method as in case $k > n$ can be used. **QED.**

Since the linear Liapunov stability excludes the algebraic growth of infinitesimal perturbations, it means that under conditions of Propositions 14 and 15 all the eigenvectors of the set F_k of stable pertubations are linear independent, and the algebraic multiplicity of the corresponding eigenvalues coincides with their geometric multiplicity.

It is now easy to show that the Legendre polynomial $P_2(\mu)$ is linearly stable. Indeed, the perturbations $\psi_1'^m(t)(m = -1, 0, 1)$ of any solution of Eq. (1) are constants. Hence, due to (39),

$$U[\psi'(t)] \equiv \sum_{k=3}^{\infty} \chi_k(\chi_k - 6) \sum_{m=-k}^{k} |\psi_k'^m(t)|^2 = U[\psi'(0)] = \text{ const}$$

for any small perturbation of the invariant linear space $F = \overset{\infty}{\underset{k=3}{\oplus}} H_k$. Therefore, $U[\psi'(t)]$ can be taken as the Liapunov functional. Since the set F is a linear space, it is not necessary to apply here Proposition 13. As a corollary we have that any R–H wave of $H_1 \oplus H_2$ is Liapunov stable in the invariant subspace F.

12. Summary

In this paper, a stability study of periodic solutions (the R–H waves and the Verkley modons) of the vorticity equation have been carried out within the framework of an ideal incompressible fluid on a rotating sphere. We have derived the conservation law for arbitrary perturbations of the R–H wave (Proposition 1) and found invariant sets of such perturbations. The conservation law and the invariant sets have also been obtained for the stationary Verkley modon, but only for small perturbations (see (121), (123)). It follows from these results that there are certain common features in the structure of small invariant perturbations of both the R–H waves and the stationary modons.

For estimating the rate of convergence or divergence of paths of the solutions of Eq. (1) in the phase space, we have introduced the metric (47) through a linear combination of the energy and the enstrophy of the perturbations. In Section 4–6, we have derived formulas which enable us to find at any moment the distance between two R–H waves, between a R–H wave and a modon or between two modons in the phase space. The formulas have been used to obtain the necessary and sufficient conditions for the distance between any two solutions from the set of all R–H waves and modons to be constant (Propositions 3, 6, 8). These formulas have been applied to prove the Liapunov instability of two types of exact solutions of the vorticity equation on a sphere: non–zonal R–H waves of the subspace $H_1 \oplus H_n$ if $n \geq 2$ and dipole modons (Propositions 11, 12).

It is shown in Section 7 that the Liapunov instability of any non-zonal R–H wave as well as dipole modons is caused by algebraic growth of perturbations. Such perturbations are the difference between the basic wave and another periodical solution such that their paths are very close to each other in phase space. However these two solutions oscillate asynchronously because of different velocities (18) or (27). This kind of instability is not connected with the orbital instability, does not exist in the linear stability problem and is typical for periodical solutions of the nonlinear conservative pendulum equation.

According to Eq. (39), the whole space of perturbations of the R–H wave from the subspace $H_1 \oplus H_n$ consists of three invariant sets M_-^n, M_o^n and M_+^n defined by the magnitude of the mean spectral number of perturbations. Therefore, the dynamics of perturbations can be analyzed in each of these sets independently. The proof of Proposition 11 shows that the R–H wave is Liapunov unstable with respect to certain perturbations from M_-^n. It is likely that the same assertion is also true for the set M_+^n, nevertheless it is still to be determined. We would like to emphasize that the energy cascades of any growing (or any decaying) perturbations have the opposite directions in M_+^n and M_-^n. As for M_o^n, it includes the invariant subset H_n, all the perturbations of which are stable (Proposition 2). The question, whether the other part of M_o^n contains unstable perturbations or not, also remains to be explored. Due to Proposition 1, the kinetic energy and enstrophy of any perturbation of the R–H wave increase, decrease or remain constant simultaneously, and hence, the interdependence between the kinetic energy and the concomitant mean spectral number of a perturbation is hyperbolic. Also note that the largest amplitudes of perturbations of the sets M_-^n and M_+^n are in the immediate proximity to the set M_o^n.

Due to Eq. (121), there is an invariant nonlinear functional for small perturbations of the Verkley stationary modon (Verkley, 1984, 1987), and a small neighbourhood of such modon can be divided into three invariant sets (123) by analogy with the R–H wave case. But unlike the law (39), the conservation law (121) is valid only for small perturbations of the stationary modon. For the particular case of Verkley (1984) (or Tribbia (1984)) stationary modons, this law coincides with that found by Laedke and Spatschek (1986) and Swaters (1986) for small perturbations of the beta–plane modon with rapidly decaying exterior solution. The conservation laws (39) and (121) and Zubov's criterion (Proposition 13) have been used in Section 11 to show that any monopole modon with $\chi_\sigma > 0$, as well as any Legendre polynomial $P_n(\mu)$, is linearly Liapunov stable with respect to special invariant sets of perturbations of sufficiently small scale (Propositions 14 and 15).

APPENDIX A. THE EULER ANGLES

As mentioned in Section 6, to bring the system $(x_1,\ y_1,\ z_1)$ into coincidence with the system $(x',\ y',\ z')$ (see Fig. 3) it is necessary to execute three successive rotations through the Euler angles ρ, $-\beta$ and ϑ, The first rotation of the system $(x_1,\ y_1,\ z_1)$ is performed about the axis $0z_1$ through the angle ρ. After that the axis $0y_1$ will coincide with the axis $0y_o$. The second rotation

is performed about the axis $0y_o$ through the negative angle $-\beta$ so as to bring into coincidence the axes $0z_1$ and $0z'$. The last rotation is carried out about the axis $0z'$ through the angle ϑ. Note that two rotations defined by matrices $D(\rho, -\beta, \vartheta)$ and $D(\pi+\rho, \beta, \pi+\vartheta)$ are equivalent (Nikiforov and Uvarov, 1984).

We now show how to find the angles ρ, β and ϑ provided the values a, b and γ are known. Since the angles N_1OB and BON' are direct, the cosine theorem (Berger, 1978) being successively applied to the spherical triangles N_1NN', N_1NB and BNN', gives

$$u = u(t) \equiv \cos\beta = ab + \sqrt{1-a^2}\,\sqrt{1-b^2}\cos\gamma \tag{A.1}$$

$$0 = a\mu_B + \sqrt{1-a^2}\,\sqrt{1-\mu_B^2}\cos\gamma_1 \tag{A.2}$$

$$0 = b\mu_B + \sqrt{1-b^2}\,\sqrt{1-\mu_B^2}\cos(\gamma-\gamma_1) \tag{A.3}$$

where γ_1 is the longitudinal angle between the points A and B and μ_B is the μ-coordinate of B. It follows from (A.2) and (A.3) that

$$tg\gamma_1 = (b\sqrt{1-a^2} - a\sqrt{1-b^2}\cos\gamma)/(a\sqrt{1-b^2}\sin\gamma) \tag{A.4}$$

$$\mu_B = -\sqrt{1-a^2}\cos\gamma_1/\sqrt{a^2+(1-a^2)\cos^2\gamma_1} \tag{A.5}$$

The cosine theorem used for the spherical triangle ANB and Eq. (A.2) gives

$$\cos\rho = -\mu_B/\sqrt{1-a^2} \tag{A.6}$$

Relating the coordinates of the point B in the systems (λ, μ) and (λ_1, μ_1) with the sine theorem (Berger, 1978) we obtain

$$\sin\rho = \sqrt{1-\mu_B^2}\sin\gamma_1 \tag{A.7}$$

The angle ρ is uniquely defined by Eqs. (A.6) and (A.7). Similar analysis for the spherical triangle $BN'C$ yields

$$\cos\vartheta = -\mu_B/\sqrt{1-b^2} \tag{A.8}$$

$$\sin\vartheta = \sqrt{1-\mu_B^2}\sin(\gamma-\gamma_1) \tag{A.9}$$

Since β is within interval $[0, \pi]$, the angle γ_1 is uniquely defined by Eq. (A.4).

APPENDIX B. THE PROOF OF PROPOSITION 9

As was mentioned before, the case $u = a^2$ corresponds to the angle $\gamma = \pi/2$, and hence,

$$h_n(u) = \frac{1}{2}(1-u)\{P_{n-1}^{(0,\,2)}(u) - P_{n-1}^{(2,\,0)}(u)\} \tag{B.1}$$

for such a value of u. Due to (87), we can rewrite (B.1) as

$$h_n(u) = v\{F(-n+1; n+2; 1; v) - \frac{\chi n}{2} F(-n+1; n+2; 3; v)\}$$

where $v = (1-u)/2$. Using the functional relation (Olver, 1974)

$$\frac{v-1}{\Gamma(s-1)} F(p; r; s-1; v) + \frac{1}{\Gamma(s)}\{s-1-(2s-p-r-1)v\} F(p; r; s; v)$$

$$+ \frac{v}{\Gamma(S+1)}(s-p)(s-r) F(p; r; s+1; v) = 0 \qquad (B.2)$$

we obtain

$$h_n(u) = F(-n+1; n+2; 1; v) - F(-n+1; n+2; 2; v)$$

We now use one more relation for hypergeometric functions from Korn and Korn (1968):

$$s(s+1)\{F(p; r; s; v) - F(p; r; s+1; v)\} - prvF(p+1; r+1; s+2; v) = 0$$

Then

$$h_n(u) = \frac{v}{2}(-n+1)(n+2) F(-(n-2); n+3; 3; v)$$

Applying again (87) we obtain

$$h_n(u) = -\frac{n+2}{2n}(1-u) P_{n-2}^{(2,\,2)}(u) \qquad (B.3)$$

Since $1-u < 1-u^2$, (B.3) leads to

$$|h_n(u)| < \frac{n+2}{2n}(1-u^2)|P_{n-2}^{(2,\,2)}(u)| \qquad (B.4)$$

Taking into account the relation

$$(1-u^2)^{m/2} P_{n-m}^{(m,\,m)}(u) = \frac{2^m n!}{(n+m)!} P_n^m(\mu) \qquad (B.5)$$

between the Jacobi polynomials and the associated Legendre functions (Nikiforov and Uvarov, 1984) we obtain

$$|h(u)| < 2\chi_n^{-1}|P_n^2(u)| \qquad (B.6)$$

The addition theorem for the spherical harmonics (Richtmyer, 1982) yields

$$\sum_{m=-n}^{n} |Q_n^m(u)|^2 = \frac{2n+1}{4\pi}$$

and hence, the inequality

$$2|Q_n^m(u)| \leq \left(\frac{2n+1}{4\pi}\right)^{1/2}$$

holds for all $m \neq 0$. Therefore, using (7)–(9), we obtain finally

$$|h_n(a^2)| \equiv |h_n(u)| < \chi_n^{-1} \left\{\frac{(n+2)!}{(n-2)!}\right\}^{1/2} < 1.$$

QED.

APPENDIX C. THE PROOF OF PROPOSITION 10

According to (34), it is sufficient to prove that

$$< J(\psi,\ \Delta\psi),\ f >= 0 \tag{C.1}$$

for any smooth function ψ, if the solution $f \epsilon H_1$. To this end, we define a projection of ψ on H_n as (see Helgason, 1984)

$$\psi_n(x) = (2n+1)(\psi * P_n)(x) \tag{C.2}$$

where the operation $(\psi * P_n)(x)$ called a convolution of the function ψ and the Legendre polynomial $P_n(\mu)$, is determined in the following way (Helgason, 1984):

$$(\psi * P_n)(x) = \frac{1}{4\pi}\int_S \psi(y) P_n(\vec{x}\cdot\vec{y})dy \tag{C.3}$$

Here x and y are points on the sphere S and $\vec{x}$ and $\vec{y}$ are the radius-vectors. The scalar product $\vec{x}\cdot\vec{y}$ is equal to $\cos\theta$ where θ is the angle between two vectors $\vec{x}$ and $\vec{y}$. The point x is taken in (C.3) as the North pole of new geographic coordinate system in which $\vec{x}\cdot\vec{y} = \cos\theta = \mu$. Note that $f(x) = 3(f * P_1)(x)$. So

$$< J(\psi,\ \Delta\psi),\ f >= 3 < J(\psi(x),\ \Delta\psi(x)),\ (f * P_1)(x) >=$$

$$= \frac{3}{4\pi} < J(\psi(x),\ \Delta\psi(x)),\ \int_S f(y) P_1(\vec{x}\cdot\vec{y})dy >=$$

$$= 3 < (J(\psi,\ \Delta\psi) * P_1)(y),\ f(y) >$$

Here we change the order of the integration over x and y and used the definition (C.3). Since $P_1(\mu) = \mu$, we obtain (C.1). Actually, for every fixed y we have

$$J(\psi, \Delta\psi) * P_1)(y) = \frac{1}{4\pi}\int_S J(\psi(\lambda,\ \mu),\ \Delta\psi(\lambda,\ \mu))\mu dS = 0$$

due to (14). **QED.**

APPENDIX D. THE PROOF OF PROPOSITION 11

We will use the formula (51). By Proposition 1, there is no loss of generality because of choice of the norm (47) as a square root of $K(t)$. Also, due to Proposition 10, suppose, without loss of generality, that $\omega = 0$, and hence, the wave (17) belongs to the subspace H_n only:

$$f(t,\ \lambda,\ \mu) = \sum_{k=-n}^{n} f_k Y_n^k(\lambda - C_n t,\ \mu), \tag{D.1}$$

Since the R–H wave (D.1) is non–zonal, $f_k \neq 0$ at least for one k satisfying $1 \leq k \leq n$. Let for definiteness, such k is odd and

$$\varepsilon = \{4\chi_n \sum_{k=1}^{p} |f_{2k-1}|^2\}^{1/2}, \tag{D.2}$$

in the definition of the Liapunov stability. Here p is the largest integer number such that $p \leq (n+1)/2$. Let an initial value of the energy of perturbations does not exceed δ^2 where δ is an arbitrary positive number, as small as we like. We now show that independently of δ and the moment t_1 chosen, there is a perturbation ψ' of the wave (D.1) and a time moment $\tau > t_1$ such that the kinetic energy $K(\tau)$ of ψ' exceeds ε^2. For the ε given by (D.2), and for any small δ chosen, we take as another solution $\psi = \psi(\delta)$ of Eq. (1) the R–H wave

$$\psi(t,\ \lambda,\ \mu) = -\frac{\delta}{2} Y_1^o(\mu) + \sum_{k=-n}^{n} f_k Y_n^k(\lambda - \hat{C}_n t,\ \mu), \tag{D.3}$$

from $H_1 \oplus H_n$ where $\hat{C}_n = \hat{C}_n(\delta) = \frac{\delta}{2} - (\delta+2)/\chi_n$. Then, according to (51), we obtain

$$K(t) = \frac{\delta^2}{4} + 2\chi_n \sum_{k=1}^{n} |f_k|^2 \{1 - \cos k(\hat{C}_n - C_n)t\} \tag{D.4}$$

for the perturbation $\psi' = \psi - f$ of the wave (D.1). Here

$$\hat{C}_n - C_n = \delta\left(\frac{1}{2} - 1/\chi_n\right). \tag{D.5}$$

If $n > 1$, then $\chi_n > 2$, and hence, $\hat{C}_n - C_n \neq 0$. Note that the case $n = 1$ (when $\hat{C}_n - C_n = 0$), confirms Proposition 10. Due to (D.4), the condition (91) is satisfied at the initial moment $t_o = 0$ since $K(0) = \frac{\delta^2}{4}$. However, if $t = \tau_j = \frac{\pi(2j+1)}{\hat{C}_n - C_n}$ (where $j = 0,\ 1,\ 2, \ldots$) then $K(\tau_j) = \frac{\delta^2}{4} + \varepsilon^2$, and the inequality (90) is false. Since the sequence $\{\tau_j\}$ tends to ∞ as $j \to \infty$, it is impossible to find such a moment $t_1 \geq t_o$ so as to satisfy the inequality (90) for all $t \geq t_1$. Therefore, by definition, the wave (D.1) (and hence, the wave (17)) is Liapunov unstable. **QED.**

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