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Second order nonlinear interactions among Rossby waves

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RESUMEN

En la teoría de interacciones de ondas de Rossby (OR) débilmente alineales, se busca una solución perturbativa para la función de corriente cuasi–geostrófica de la forma $\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \ldots$, donde ε es el número de β–Rossby. Si la solución a orden cero $\psi^{(0)}$ es *cualquier* superposición de OR, la ecuación de perturbación a segundo orden para $\psi^{(2)}$ siempre tiene un forzamiento resonante. El hecho de que a $O(\varepsilon^2)$ siempre haya resonancia se contrasta con interacciones alineales a primer orden, donde las tríadas resonantes forman un conjunto muy restringido entre todas las posibles interacciones. Como ejemplo se toma a $\psi^{(0)}$ igual a la suma de dos OR arbitrarias en un océano sin fronteras laterales. A segundo orden, el método de escalas de tiempo múltiples da lugar a un efecto Doppler de $O(\varepsilon^2)$ en la frecuencia de cada onda proporcional al cuadrado de la amplitud de la otra onda. Usando parámetros realistas para la onda-1 y la amplitud de la onda-2, se encuentra que el cambio en la frecuencia no es despreciable en regiones del plano de número de onda de la onda-2 cercanas a resonancia a primer orden. Es entonces concebible tener un campo de OR, débilmente interactuando, tal que a $O(\varepsilon)$ no haya interacciones resonantes; sin embargo, el efecto Doppler en sus frecuencias, aunque pequeño, *siempre* tendrá lugar, debido a que siempre hay resonancia a $O(\varepsilon^2)$.

ABSTRACT

In the weakly nonlinear interaction theory of mid–latitude Rossby waves (RWs), a perturbative solution for the quasi–geostrophic (QG) streamfunction is sought in the form $\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \ldots$, where ε is the β–Rossby number. If the leading order solution $\psi^{(0)}$ is *any* superposition of RWs, the second order perturbation equation for $\psi^{(2)}$ always has resonant forcing. This is in contrast to first order nonlinear interactions where resonant triads form a very restricted set of all possible interactions. An example is worked out in a laterally unbounded ocean and taking $\psi^{(0)}$ as the superposition of two arbitrary RWs. At second order, multiple time scales lead to an $O(\varepsilon^2)$ Doppler shift of the frequency of each wave, proportional to the amplitude squared of the other one. Using realistic wave parameters for wave-1 and the amplitude of wave-2, the frequency shift is not negligible in regions of wavenumber space of wave-2 near resonance at first order. It is thus conceivable to have a field of weakly interacting RWs, such that at $O(\varepsilon)$ there are no resonant interactions; however the Doppler shift in their frequencies, albeit small, will *always* take place due to resonance at $O(\varepsilon^2)$.

1. Introduction

Waves occur in Nature in an astonishing variety of physical systems. Linear theories of wave motion were developed during the XIX and XX centuries to a high degree of sophistication, particularly in hydrodynamics (Craik, 1985). Since the governing equations and boundary conditions of fluid mechanical systems in general are nonlinear, the linearized approximation restricts attention to sufficiently small wave amplitudes. But even during the XIX century, progress was made in understanding some aspects of weakly–nonlinear wave propagation, in which linear theory is considered to provide a good starting point in the search for better, higher-order, approximations. A celebrated accomplishment was that of Stokes (1847) for surface gravity waves.

In weakly-nonlinear wave theories there is great interest in studying resonant interactions, mainly for two reasons. First, all nonresonant interactions only produce a small amplitude background noise of forced waves whose amplitudes are small compared to those waves produced by the resonant interactions (Podlosky, 1987), although there is experimental evidence that dynamically non-resonant interactions might be important in surface gravity waves (Hammack and Henderson, 1993). Second, unlike nonresonant interactions, resonant interactions can cause significant energy transfer among waves and profoundly affect wavefield evolution.

The theory of resonant interactions among planetary Rossby waves (RWs) has had a long history. Longuet-Higgins and Gill (1967) studied resonant interactions of barotropic, divergent (free surface) RWs in a laterally unbounded ocean. Resonant triads are possible and all wave vectors can participate in a resonant triad with a family of wave vectors. If the RWs are baroclinic, account must be taken for the nonlinear coupling between vertical normal modes. In Graef-Ziehl (1990), the nonlinear interaction between an incident and the reflected RW at a straight coast was studied. Resonance at first order (Graef, 1993), in which the incident, reflected and forced RW (with frequency twice that of the incident wave) form a resonant triad, is severely limited. There could be resonance only if $0 < |\sin\alpha| \leq 1/3$, where α is the angle between the reflecting wall and the eastern direction. In such case, the wave amplitudes are slowly varying periodic functions of the offshore coordinate. At the next order, i.e. at $O(\varepsilon^2)$, where ε is the β-Rossby number, there is *always* resonance (Graef-Ziehl, 1990), which leads to a shift in the offshore wavenumbers; the wave amplitudes remain constant.

The motivation behind this work is to answer the questions: Does resonance at $O(\varepsilon^2)$ happen in the RW reflection problem only, i.e. having as primary waves an incident and the reflected RW? What happens at $O(\varepsilon^2)$ if there are no lateral boundaries? It will be shown that the answer to the first question is negative, by proving that if the leading order solution, $\psi^{(0)}$, is a superposition of two arbitrary Rossby waves, then the second order perturbation equation for $\psi^{(2)}$ always has resonant forcing. The method of multiple scales is then used to remove the secular terms from the forcing of $\psi^{(2)}$, which leads to an $O(\varepsilon^2)$ shift of the frequency of each wave, proportional to the amplitude squared of the other one. This frequency shift is reminiscent of a widely known nonlinear effect on surface gravity waves, namely that the frequency exceeds that of linear theory (see e.g. Lamb, 1932), or to be more precise, that in Stokes wave the phase speed depends on the square of its amplitude.

This paper is organized as follows. In Section 2 the solution up to first order in ε is given. The appearance of resonant forcing at $O(\varepsilon^2)$ is shown in Section 3, where the method of multiple scales is used to show the frequency shift, and the results are illustrated with some examples. A discussion and conclusions are presented in Section 4. In the appendix, an easy graphical method is given to find resonant triads at first order in the case where two members of the triad have the same frequency.

2. Solution up to first order

The coordinate system is cartesian with x eastwards, y northwards and z vertically upwards. The governing equation is the quasigeostrophic potential vorticity equation (QGPVE) (see e.g. Pedlosky, 1987). In the weakly nonlinear interaction theory of mid-latitude Rossby waves, a perturbative solution for the quasigeostrophic (QG) streamfunction is sought in the form

$$\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \dots \tag{2.1}$$

We consider the case where the ocean is laterally unbounded, and for simplicity, we restrict the waves to be barotropic. Then, let $\psi^{(0)}$ be the superposition of any two barotropic RWs:

$$\begin{aligned}\psi^{(0)} &= A_1 \cos(\vec{k}_1 \cdot \vec{x} - \omega_1 t + \phi_1) + A_2 \cos(\vec{k}_2 \cdot \vec{x} - \omega_2 t + \phi_2)\\ &\equiv A_1 \cos\Theta_1 + A_2 \cos\Theta_2,\end{aligned} \tag{2.2}$$

where $\vec{x} = (x,\ y)$ and each RW satisfies the dispersion relation, viz.

$$\omega_i = \sigma(\vec{k}_i) \equiv \frac{-\beta\vec{k}_i \cdot \hat{\imath}_E}{|\ \vec{k}_i\ |^2 + r_e^{-2}},\ i = 1,\ 2, \tag{2.3}$$

in which $\hat{\imath}_E$ is the unit vector in the eastward direction, β is the northward gradient of the planetary vorticity and $r_e = (gH)^{\frac{1}{2}}/f_o$ is the barotropic or external Rossby radius of deformation, where g is the acceleration of gravity, H is the water depth and f_o is the Coriolis parameter. Typical values of r_e are in the range 2000 km to 4000 km in the open ocean.

Although each RW is an exact solution of the nonlinear QGPVE, the sum will not be, in general, a nonlinear solution. The nonlinear interaction of these two waves will produce a forcing in the problem for $\psi^{(1)}$ wich oscillates with the sum and difference of their two phases, i.e.

$$\begin{aligned}J\left(\psi^{(0)},\ \nabla^2\psi^{(0)}\right) &= -A_1A_2\left(|\ \vec{k}_2\ |^2 - |\ \vec{k}_1\ |^2\right)\hat{k}\cdot\vec{k}_1 \times \vec{k}_2 \sin\Theta_1 \sin\Theta_2\\ &= -A_1A_2\Gamma_{12}[\cos(\Theta_1 - \Theta_2) - \cos(\Theta_1 + \Theta_2)],\end{aligned} \tag{2.4}$$

where $J(A,\ B) \equiv (\partial_x A)(\partial_y B) - (\partial_y A)(\partial_x B)$ is the Jacobian operator, $\hat{k}$ is the vertical unit vector and $\Gamma_{12} \equiv \frac{1}{2}\left(|\ \vec{k}_2\ |^2 - |\ \vec{k}_1\ |^2\right)\hat{k}\cdot\vec{k}_1 \times \vec{k}_2$ is the coupling coefficient. For completeness, if the waves are baroclinic, i.e. if

$$\psi^{(0)} = A_1\Psi_{n_1}(z)\cos\Theta_1 + A_2\Psi_{n_2}(z)\cos\Theta_2, \tag{2.5}$$

where $\Psi_{n_1}(z)$ and $\Psi_{n_2}(z)$ are eigenfunctions of the vertical Sturm–Liouville problem with eigenvalues λ_{n_1} and λ_{n_2} and $\omega_i = \sigma_{n_i}(\vec{k}_i) \equiv -\beta\vec{k}_i \cdot \hat{\imath}_E / \left(|\ \vec{k}_i\ |^2 + f_o^2\lambda_{n_i}\right)$, $i = 1,\ 2$, then the nonlinear interaction of these two waves is

$$\begin{aligned}&J\left\{\psi^{(0)},\ \nabla^2\psi^{(0)} + \partial_z\left[\frac{f_o^2}{N^2(z)}\partial_z\psi^{(0)}\right]\right\} =\\ &-A_1A_2\Gamma_{12}^{n_1n_2}\Psi_{n_1}\Psi_{n_2}[\cos(\Theta_1 - \Theta_2) - \cos(\Theta_1 + \Theta_2)],\end{aligned} \tag{2.6}$$

where $\Gamma_{12}^{n_1n_2} \equiv \frac{1}{2}\left(|\ \vec{k}_2\ |^2 + f_o^2\lambda_{n_2} - |\ \vec{k}_1\ |^2 - f_o^2\lambda_{n_1}\right)\hat{k}\cdot\vec{k}_1 \times \vec{k}_2$ and $N(z)$ is the Brunt–Väisälä frequency. $\Gamma_{12}^{n_1 n_2}$ is symmetric in the indices 1 and 2, but it is not in $\vec{k}_{1,\ 2}$ nor in $n_{1,\ 2}$ individually. Therefore, two Rossby waves will not interact if their coupling coefficient vanishes, i.e. if either

1. Their wavenumber vectors are parallel ($\vec{k}_1 \times \vec{k}_2 = \vec{0}$).
2. The waves have the same vertical mode (i.e. same vertical structure) *and* the same wavelength.
3. The waves have the same total wavenumber, i.e. the same $|\ \vec{k}\ |^2 + f_o^2 \lambda_n$. From the dispersion relation this is equivalent to say that the waves have the same zonal or x–slowness (Ripa, 1981), which is $\vec{k} \cdot \hat{i}_E / \omega$.

Obviously condition 2 $\Rightarrow$ condition 3 but 3 $\not\Rightarrow$ 2 so condition 2 is more restrictive. Notice that in condition 1 each advection is zero independently and then their sum is zero. In contrast, when condition 2 or 3 occurs, it is the *sum* of the two advections which exactly cancel each other giving zero interaction. It is difficult in these cases 2 or 3 to see geometrically why there is no interaction. The discussion just presented for baroclinic waves complements that in Pedlosky (1987) for barotropic waves.

The first order $[O(\varepsilon)]$ perturbation QGPVE is

$$\partial_t \left(\nabla^2 \psi^{(1)} - r_e^{-2} \psi^{(1)} \right) + \beta \partial_x \psi^{(1)} = A_1 A_2 \Gamma_{12} [\cos(\Theta_1 - \Theta_2) - \cos(\Theta_1 + \Theta_2)]. \tag{2.7}$$

A particular forced solution for $\psi^{(1)}$ is

$$\psi^{(1)} = B_d^{(1)} \sin(\Theta_1 - \Theta_2) + B_s^{(1)} \sin(\Theta_1 + \Theta_2), \tag{2.8}$$

where

$$B_d^{(1)} = \frac{-A_1 A_2 \Gamma_{12}}{[\sigma(\vec{k}_1 - \vec{k}_2) - (\omega_1 - \omega_2)](|\ \vec{k}_1 - \vec{k}_2\ |^2 + r_e^{-2})}, \tag{2.9}$$

$$B_s^{(1)} = \frac{A_1 A_2 \Gamma_{12}}{[\sigma(\vec{k}_1 + \vec{k}_2) - (\omega_1 + \omega_2)](|\ \vec{k}_1 + \vec{k}_2\ |^2 + r_e^{-2})}, \tag{2.10}$$

unless there is resonance, i.e. unless the frequency of any of the forced waves coincides with the frequency of a free, linear wave, i.e. unless $\omega_1 - \omega_2 = \sigma(\vec{k}_1 - \vec{k}_2)$ or $\omega_1 + \omega_2 = \sigma(\vec{k}_1 + \vec{k}_2)$. The resonant case at this order has been extensively studied (Longuet–Higgins and Gill, 1967; Ripa, 1981; Pedlosky, 1987). The problem of finding resonant triads in the particular case $\omega_1 = \omega_2$, for which only the term $\sim \cos(\Theta_1 + \Theta_2)$ could be resonant, is examined in the appendix.

We now *insist* that wave-1, wave-2 and either of the two forcing terms do *not* form a resonant triad. This is usually the case since for a given wave the geometric locus of the other two wave vectors participating in a resonant triad is a curve. In other words, if one chooses two arbitrary RWs at random, chances are extremely high that neither of the forced waves is going to be a free RW. It should be kept in mind that philosophically, it is generally assumed that the *lowest*–order resonant interactions that occur will dominate wavefield evolution. In principle, this mitigates interest in higher–order effects. Also in practice, the labor involved in the dynamical calculation of higher–order resonances is daunting. However, it is the issue of this paper to show that second order resonance always occur, leading to a shift in the frequency of the primary waves.

3. Second order: resonant forcing

The second order problem for $\psi^{(2)}$ has the following forcing, produced by the nonlinear interaction of $\psi^{(0)}$ and $\psi^{(1)}$:

$$J\left(\psi^{(0)},\ \nabla^2\psi^{(1)}\right)+J\left(\psi^{(1)},\ \nabla^2\psi^{(0)}\right)=$$

$$f_1\sin\Theta_1+f_2\sin\Theta_2+f_3\sin(2\Theta_1-\Theta_2)+$$

$$f_4\sin(2\Theta_2-\Theta_1)+f_5\sin(2\Theta_1+\Theta_2)+f_6\sin(2\Theta_2+\Theta_1), \tag{3.1}$$

where the first two coefficients are

$$f_1=\frac{1}{2}A_2\hat{k}\cdot\vec{k}_1\times\vec{k}_2\left[B_d^{(1)}\left(2\vec{k}_1\cdot\vec{k}_2-\mid\vec{k}_1\mid^2\right)+B_s^{(1)}\left(2\vec{k}_1\cdot\vec{k}_2+\mid\vec{k}_1\mid^2\right)\right], \tag{3.2}$$

$$f_2=\frac{1}{2}A_1\hat{k}\cdot\vec{k}_1\times\vec{k}_2\left[B_d^{(1)}\left(2\vec{k}_1\cdot\vec{k}_2-\mid\vec{k}_2\mid^2\right)-B_s^{(1)}\left(2\vec{k}_1\cdot\vec{k}_2+\mid\vec{k}_2\mid^2\right)\right]. \tag{3.3}$$

Clearly, the nonlinear interaction of $\psi^{(0)}$ and $\psi^{(1)}$ has produced two resonant forcing terms (unless their coefficients f_1, f_2 vanish; in general they do not). The terms are resonant because they are themselves free RWs or homogeneous solutions of (2.7). Except for their amplitude and a phase shift of $\pi/2$ (from cosine to sine), the resonant terms are precisely the two primary waves ($\psi^{(0)}$).

A particular solution to the resonant forcing terms grows linearly in time and is called secular[1]. A uniformly valid solution (in time) to $O(\varepsilon^2)$ is sought using the method of multiple time scales (Bender and Orszag, 1978; Nayfeh, 1981). A new (slow) time scale is introduced: $T_2=\varepsilon^2 t$. There is no need to introduce $T_1=\varepsilon t$, since it was assumed that there are no resonant interactions at first order. The straightforward or pedestrian expansion solution that grows linearly in $\varepsilon^2 t$ suggests that the amplitudes and phases of the primary RWs be functions of T_2, so that the leading order solution is written now as

$$\psi^{(0)}=A_1(T_2)\cos\Theta_1+A_2(T_2)\cos\Theta_2, \tag{3.4}$$

where now $\Theta_i\equiv\vec{k}_i\cdot\vec{x}-\omega_i t+\phi_i(T_2)$, $i=1,\ 2$, and the QGPVE at $O(\varepsilon^2)$ has the following additional term on its right hand side:

$$-\partial_{T_2}\left(\nabla^2-r_e^{-2}\right)\psi^{(0)}=$$

$$\left(\mid\vec{k}_1\mid^2+r_e^{-2}\right)(\cos\Theta_1\partial_{T_2}A_1-A_1\sin\Theta_1\partial_{T_2}\phi_1)+$$

$$\left(\mid\vec{k}_2\mid^2+r_e^{-2}\right)(\cos\Theta_2\partial_{T_2}A_2-A_2\sin\Theta_2\partial_{T_2}\phi_2). \tag{3.5}$$

[1] The word secular is derived from the French word siécle, which means a century.

The elimination of secular terms is achieved by setting:

$$\left.\begin{aligned} \partial_{T_2} A_1 = \partial_{T_2} A_2 = 0 \\ A_1 \left(| \vec{k}_1 |^2 + r_e^{-2} \right) \partial_{T_2} \phi_1 + f_1 = 0 \\ A_2 \left(| \vec{k}_2 |^2 + r_e^{-2} \right) \partial_{T_2} \phi_2 + f_2 = 0, \end{aligned}\right\} \tag{3.6}$$

whose solution is

$$\left.\begin{aligned} A_1 &= A_1(t=0) = A_{10} \\ A_2 &= A_2(t=0) = A_{20} \\ \phi_1 &= \frac{-f_1}{A_1(| \vec{k}_1 |^2 + r_e^{-2})} T_2 + \phi_{10} \\ \phi_2 &= \frac{-f_2}{A_2(| \vec{k}_2 |^2 + r_e^{-2})} T_2 + \phi_{20}, \end{aligned}\right\} \tag{3.7}$$

where $\phi_{io} = \phi_i(t=0),\ i = 1,\ 2$ are the initial values of the phases.

Thus, resonant interactions at $O(\varepsilon^2)$ do not change the amplitude of the waves, but only their phases. As far as their dependence on T_2 is concerned, the amplitudes are constants and the phases are linear in T_2. From (3.2), (3.3), (2.9) and (2.10) it follows that $f_1 \sim A_1 A_2^2$ and $f_2 \sim A_1^2 A_2$, implying that

$$\phi_1 \sim A_2^2 T_2 \quad \text{and} \quad \phi_2 \sim A_1^2 T_2. \tag{3.8}$$

Due to resonance, second order nonlinear interactions lead to an $O(\varepsilon^2)$ correction in the frequency of wave-1 proportional to the amplitude squared of wave-2, and vice versa, for we can write:

$$\begin{aligned} \psi^{(0)} = & A_{10} \cos \left\{ \vec{k}_1 \cdot \vec{x} - \left[\omega_1 + f_1 A_1^{-1} \left(| \vec{k}_1 |^2 + r_e^{-2} \right)^{-1} \right] t + \phi_{10} \right\} + \\ & A_{20} \cos \left\{ \vec{k}_2 \cdot \vec{x} - \left[\omega_2 + f_2 A_2^{-1} \left(| \vec{k}_2 |^2 + r_e^{-2} \right)^{-1} \right] t + \phi_{20} \right\}. \end{aligned} \tag{3.9}$$

The reader might have noticed the absence of the factor ε^2 multiplying the frequency corrections. The reason is that dimensional variables are being used and in such case ε plays the role of an ordering parameter, which is set equals to one after doing the calculations. In other words, the expressions for the phases $\phi_{1,\ 2}$ are given by (3.7) with T_2 replaced by the dimensional time, t, when dimensional variables are used. For example, the perturbative solution is $\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \dots$. On the other hand, if non-dimensional variables are used, $\phi_{1,\ 2}$ would be given by (3.7) with $T_2 = \varepsilon^2 t,\ \varepsilon = U/(\beta L^2)$, in which L and U are horizontal length and horizontal velocity scales, respectively, and t would be the non-dimensional time; but obviously, when dimensional variables are restored, all scales would drop out. Although formally one should work with non-dimensional variables when using a perturbation expansion and multiple scales (Nayfeh, 1981), this may be confusing [e.g. β would not appear in the dispersion relation (2.3)]. The validity of the expansion for a given set of wave parameters (including their amplitude) may be checked ***a posteriori***, by comparing for example $\psi^{(1)}$ to $\psi^{(0)}$ in some integral sense (Graef and Magaard, 1993).

As mentioned in the introduction, a similar effect of a frequency shift occurs in nonlinear surface gravity waves where the phase speed (or frequency) depends on the square of the wave amplitude. The frequency shift in a Stokes wave can only be found if the frequency is also expanded in terms of the small parameter of the problem, the wave steepness ka, where k is the magnitude of the wavenumber vector and a is the wave amplitude; or, equivalently, if multiple time scales are introduced in the perturbation expansion. The elimination of a secular term at order $(ka)^2$ leads to the frequency shift. There is, however, a fundamental difference: whereas the self interaction of a RW is zero, that of a surface gravity wave is not. This is the reason why in the Stokes expansion one can have a single wave train but at least two primary RWs in the leading order solution of ψ.

In a more general framework, the frequency shift has to do with the broadening of the dispersion relation caused by nonlinear wave–wave interactions (Peter Muller, personal communication).

The advantages of an analytical solution can now be exploited by computing in a straightforward way the correction to the frequencies as a function of different wave parameters, without actually computing the solution. Given wave-1 (i.e. given $\vec{k}_1$ and A_1) and A_2, one can compute the maximum horizontal particle speed at first order of the forced terms, $U_d^{(1)} = |B_d^{(1)}||\vec{k}_1 - \vec{k}_2|$ and $U_s^{(1)} = |B_s^{(1)}||\vec{k}_1 + \vec{k}_2|$, and the frequency corrections of both waves as a function of the wavelength of wave-2, Λ_2, and the angle that $\vec{k}_2$ makes with the eastern direction (positive anticlockwise), θ_2, as shown in figures 1-2. Note that if the correct frequencies are written as $\omega_i + \delta\omega_i$, then the corrected periods are $T_i[1 - \delta\omega_i/\omega_i + O(\delta\omega_i/\omega_i)^2]$, where $T_i = 2\pi/\omega_i$ and assuming $|\delta\omega_i|/\omega_i < 1$.

The first case (Fig. 1), in which wave-1 is an annual period first baroclinic mode[2] with a wavelength of 315 km and $\theta_1 = 120^\circ$, shows regions in (Λ_2, θ_2)–space where both $U_d^{(1)}$ and $U_s^{(1)}$ exceed the corresponding particle speeds of the primary waves. There the perturbative solution is not valid, and correspond to waves-2 near resonance at first order, where wave-1, wave-2 and one of the two forced waves, $\sim B_d^{(1)}$ or $\sim B_s^{(1)}$, form a resonant triad. As resonance is approached, $|B_d^{(1)}| \to \infty$ or $|B_s^{(1)}| \to \infty$ and $|\delta\omega_{1,2}| \to \infty$. The maximum frequency shift in the permissible region, say where both $U_d^{(1)}$ and $U_s^{(1)}$ are less than 1 cm/s or half the particle speed of the primary waves, is in the upper and lower left corners with values of $|\delta\omega_i|/\omega_i$ of up to 25%, corresponding to periods of 2 to 4 years of wave-2. The rest of the permissible region shows values of $|\delta\omega_i|/\omega_i$ less than 5%. The β–Rossby number of wave-1 is $\varepsilon_1 = U_1|\vec{k}_1|^2/\beta = 0.4$ and $\varepsilon_2 < 1$ for $\Lambda_2 > 200$ km.

The second example (Fig. 2) has wave-1 as in the first case but with its phase due west ($\theta_1 = 180^\circ$) as well as its energy, giving $\Lambda_1 = 984$ km and $\varepsilon_1 = 0.04$, a nonlinearly weaker wave. The permissible region shows that $|\delta\omega_i|/\omega_i < 1\%$ and regions of $|\delta\omega_i|/\omega_i > 10\%$ correspond to waves-2 in or near resonance with wave-1 and the forced wave $\sim B_d^{(1)}$.

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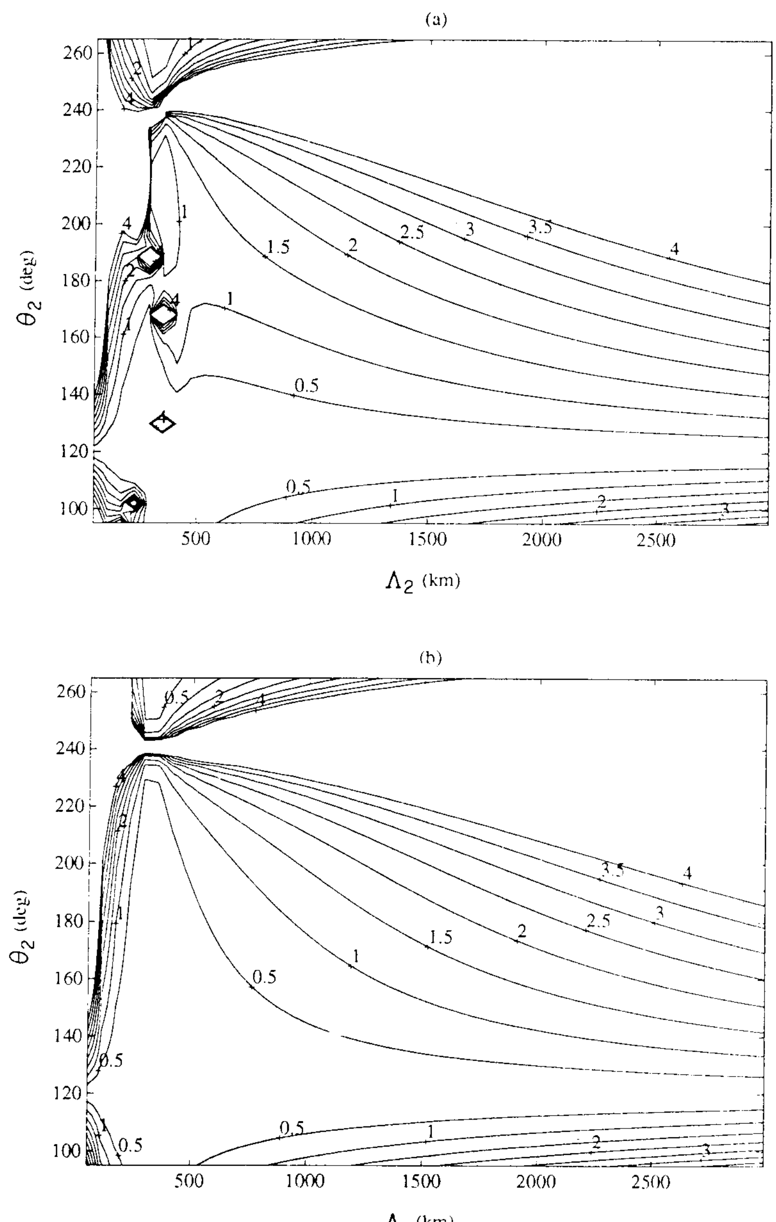
[2] Neglecting the nonlinear coupling between this mode and that of wave-2, which is also first mode, or taking the point of view of a barotropic ocean with an equivalent depth corresponding to a first mode baroclinic Rossby radius of deformation.

The first example shows that the frequency corrections are not negligible in regions of (Λ_2, θ_2)–space sufficiently close to resonance at first order but far enough such that the perturbative solution be valid. In general, it would be fair to say that outside regions of wavenumber space of wave-2 near resonance at $O(\varepsilon)$, the frequency correction of the primary waves due to resonance at $O(\varepsilon^2)$ is negligible.



(Fig. 1)

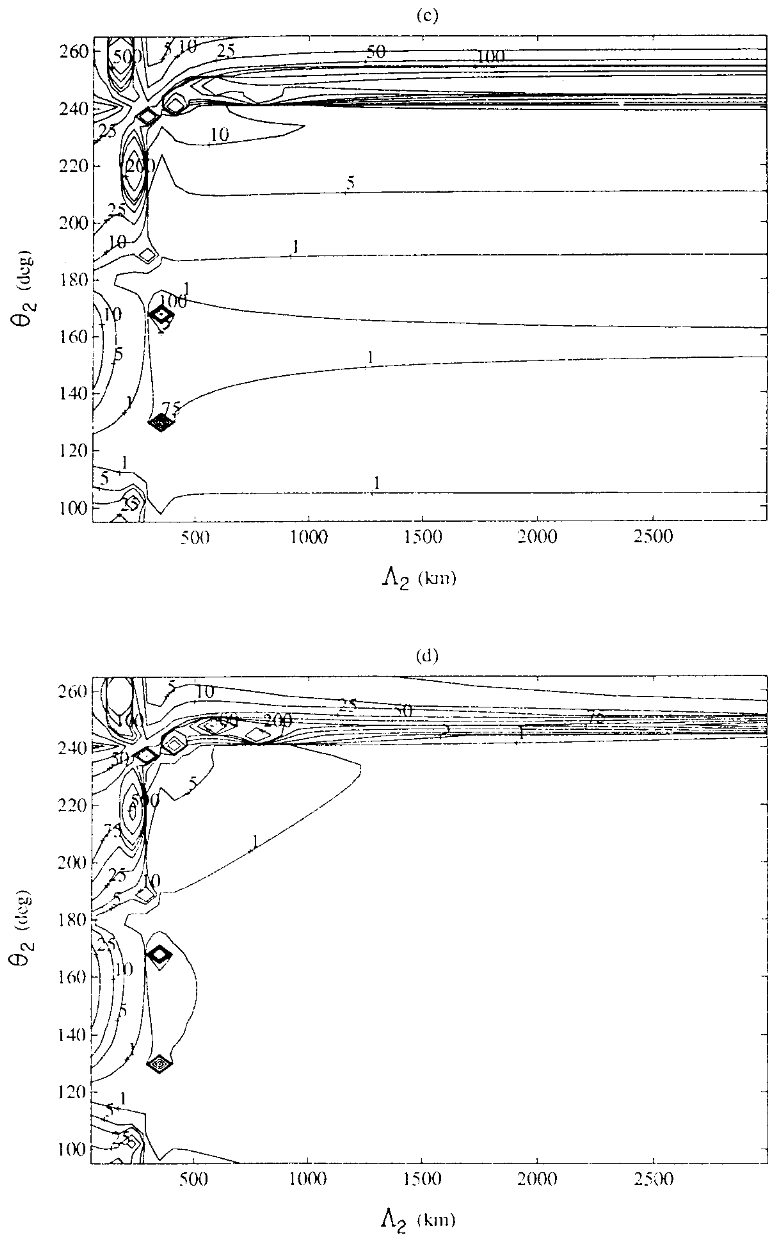
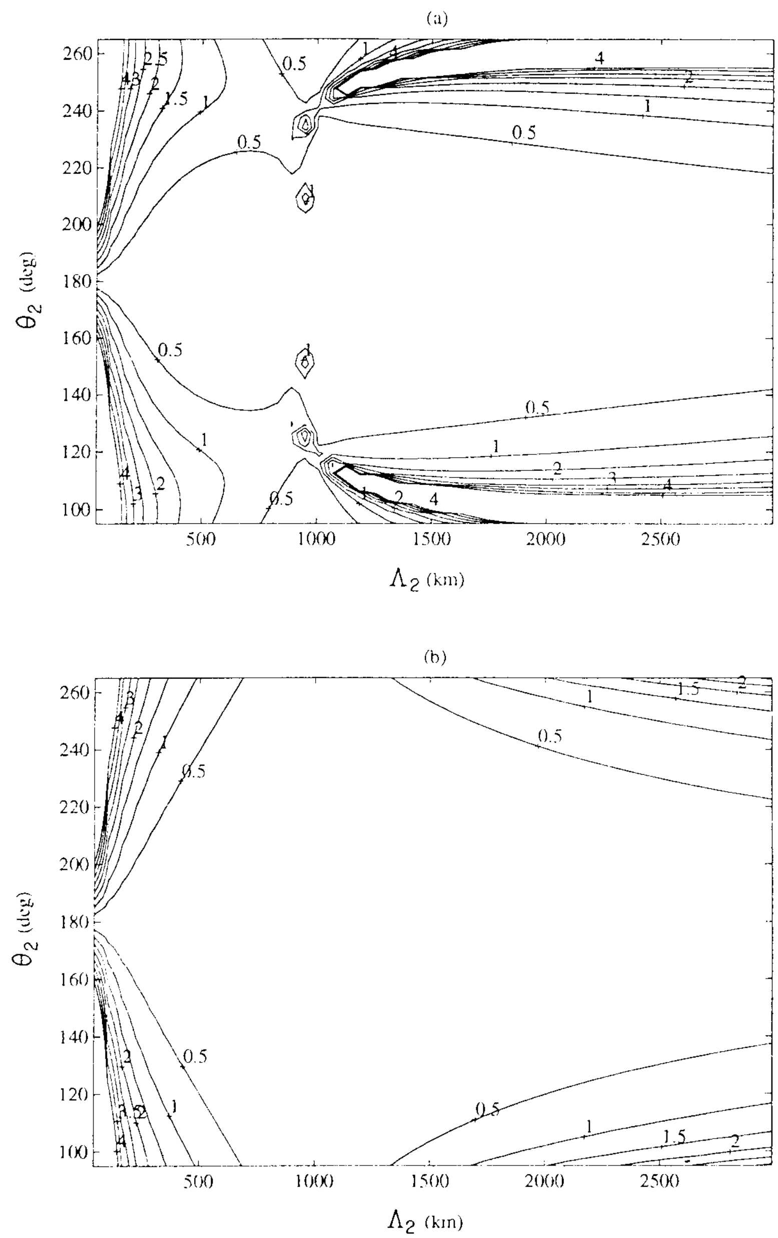


Figure 1. Maps of (a) the maximum horizontal particle speed at first order of the forced term $U_d^{(1)} = | B_d^{(1)} || \vec{k}_1 - \vec{k}_2 |$ in cm/s; (b) idem for $U_s^{(1)} = | B_s^{(1)} || \vec{k}_1 + \vec{k}_2 |$; (c) the frequency correction of wave–1, $| \delta\omega_1 | / \omega_1$ in % and (d) idem for wave–2, $| \delta\omega_2 | / \omega_2$ in % as functions of the wavelength of wave–2, Λ_2, and the angle that $\vec{k}_2$ makes with the eastern direction (positive anticlockwise), θ_2 . Reference latitude = 30°, vertical mode number of both waves $n = 1$, baroclinic Rossby radius of deformation = 41 km, period of wave–1 = 1 year, $\theta_1 = 120°$, wavelength of wave–1 $\Lambda_1 = 315$ km, maximum horizontal particle speed of primary waves $U_1 = U_2 = 2$ cm/s, β–Rossby number of wave–1 $\varepsilon_1 = U_1 |\vec{k}_1|^2 / \beta = 0.4$. Note the correspondence between the high values of $U_d^{(1)}$ and $U_s^{(1)}$ and those of $| \delta\omega_i | / \omega_i$.



(Fig. 2)

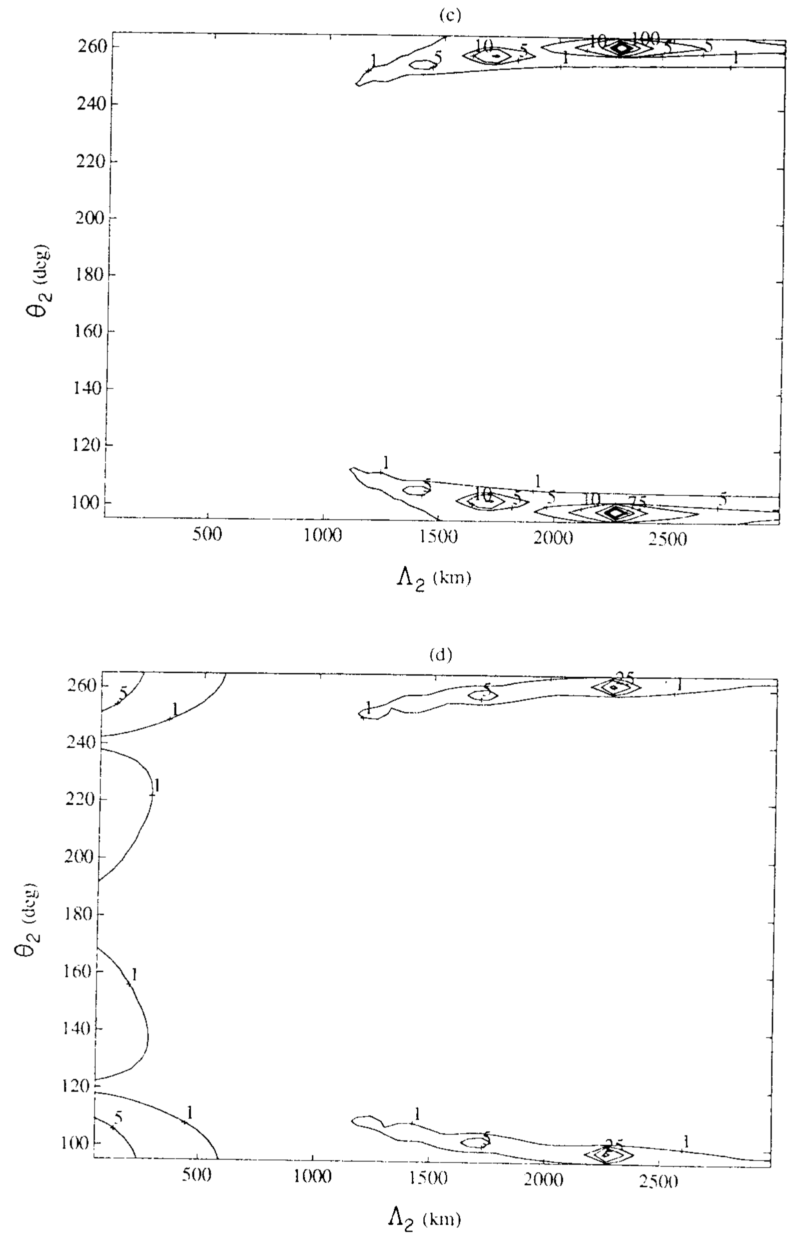


Figure 2. As in Figure 1, except that $\theta_1 = 180°$, $\Lambda_1 = 984$ km, $\varepsilon_1 = U_1 \mid \vec{k}_1 \mid^2 / \beta = 0.04$. Note the low values (less than 1%) of $\mid \delta\omega_i \mid / \omega_i$ over most of the region and high values in regions where wave-2 is in or near resonance with wave-1 and the forced wave $\sim B_d^{(1)}$.

4. Discussion and conclusions

It was shown that if the leading order solution of the QGPVE is the superposition of two arbitrary RWs, then the second order perturbation equation for $\psi^{(2)}$ always has resonant forcing, which is similar to $\psi^{(0)}$, but with the RWs having other amplitude and a phase change of 90°. Generalization of this result to a superposition of an arbitrary number of RWs follows immediately. In contrast to first order nonlinear interactions where one has to search for resonant triads, resonant forcing at second order is the rule rather than the exception.

The occurrence of resonant forcing at second order is due to the fact that the governing equation, being the QGPVE, has a quadratic nonlinearity. This can be easily understood by considering a nonlinear, energy conserving dynamical system governed by $\mathcal{N}[\varphi] = 0$, where $\mathcal{N}$ is a nonlinear operator and φ a solution. Linearizing this equation, one obtains $\mathcal{L}[\varphi_o] = 0$, where $\mathcal{L}$ is a linear operator and φ_o a (linear) solution. Writing some sort of expansion for φ, i.e. $\varphi = \sum_{j=o}^{\infty} \varepsilon^j \varphi_j$ and assuming $\mathcal{L}$ has constant coefficients, one gets

$$\left.\begin{aligned} \varphi_o &\sim \cos\theta \\ \mathcal{L}[\varphi_1] = \mathcal{Q}[\varphi_o] &\sim \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \\ \mathcal{L}[\varphi_2] = \mathcal{Q}[\varphi_0,\ \varphi_1] &\sim \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta, \end{aligned}\right\} \tag{4.1}$$

where $\mathcal{Q}$ is a quadratic operator and θ a wave phase. The equation for φ_2 has a forcing that is a homogeneous solution of the linear equation, thus it is resonant.

Multiple scales were used to avoid secular terms in $\psi^{(2)}$, which leads to a change in the *phase* of the primary waves, whereas the amplitudes remain constant. The phase change can be interpreted as a frequency correction of each wave that results proportional to the amplitude squared of the other one. In the reflection problem, where the two primary waves are not arbitrary but have a relationship, resonance at second order leads to a correction in the offshore wavenumbers of the incident–reflected waves (Graef–Ziehl, 1990). The problem here can be thought of as homogeneous in space and inhomogeneous in time, as an initial value problem. The reflection problem is homogeneous in time but inhomogeneous in space (in the offshore direction). Thus, in both problems, there is a correction in the Fourier space variable of the inhomogeneity.

Another example where nonlinearities change the frequency is a nonlinear pendulum with cubic nonlinearity, whose evolution is governed by the so called Duffing equation. Such pendulum has its period reduced by the nonlinearity (Nayfeh, 1981).

The frequency shift in the primary waves does not imply that there is an instability, as it does in the case of resonance at first order (Pedlosky, 1987). Thus, in principle we could conceive a field of RWs weakly interacting with each other, such that at $O(\varepsilon)$ there are no resonant interactions; however the shift in their frequencies will always be present due to resonance at $O(\varepsilon^2)$.

The theory in this paper has been given in terms of oceanographic parameters. In principle, it would be straightforward to present the theory in terms of atmospheric parameters. The wave parameters would of course have different numerical values, and the vertical structure functions would be different, but the same results should hold.

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APPENDIX

Resonant triads in the case $\omega_1 = \omega_2$

Here I develop a very simple graphical method to find resonant triads of barotropic RWs at $O(\varepsilon)$ when two of the waves have the same frequency. Let

$$\omega_1 = \omega_2 = \omega, \tag{A.1}$$

and let $(k,\ l)$ be the $(x,\ y)$ components of $\vec{k}$. Then, only the term $\sim \cos(\Theta_1 + \Theta_2)$ in (2.7) could be resonant, for the other term, whose response is $B_d^{(1)} \sin(\Theta_1 - \Theta_2)$, would require $\sigma(\vec{k}_1 - \vec{k}_2) = \omega_1 - \omega_2 = 0$, which implies $k_1 = k_2$ so that the waves have the same zonal or x–slowness and therefore $\Gamma_{12}^{n_1 n_2} = 0$, i.e the waves would not interact.

The problem is then to investigate whether $\sigma(\vec{k}_1 + \vec{k}_2) = 2\omega$. There are five variables: $\vec{k}_1$, $\vec{k}_2$ and ω, and three equations: $\omega = \sigma(\vec{k}_1)$, $\omega = \sigma(\vec{k}_2)$ and $2\omega = \sigma(\vec{k}_1 + \vec{k}_2)$. Thus, there are two degrees of freedom. Combining the first two equations yields

$$\frac{k_1}{|\vec{k}_1|^2 + r_e^{-2}} = \frac{k_2}{|\vec{k}_2|^2 + r_e^{-2}}, \tag{A.2}$$

whereas the third equation is

$$\frac{k_1 + k_2}{|\vec{k}_1|^2 + |\vec{k}_2|^2 + 2\vec{k}_1 \cdot \vec{k}_2 + r_e^{-2}} = \frac{2k_2}{|\vec{k}_2|^2 + r_e^{-2}}. \tag{A.3}$$

Manipulation of (A.3) and using (A.2) results in

$$|\vec{k}_1|^2 + |\vec{k}_2|^2 + 4\vec{k}_1 \cdot \vec{k}_2 = 0, \tag{A.4}$$

after division by $k_2 \neq 0$ (if $k_2 = 0$ then $k_1 = 0$ so $\vec{k}_1 \| \vec{k}_2$ and $\Gamma_{12} = 0$). Thus $\vec{k}_1 \cdot \vec{k}_2 < 0$ to have a solution and the angle between $\vec{k}_1$ and $\vec{k}_2$ is in the interval $(\pi/2,\ \pi]$.

Given $\vec{k}_1 = (k_1,\ l_1)$, the wave vectors $\vec{k}_2 = (k_2,\ l_2)$ which are solutions of (A.4) must lie on a circle of radius $3^{\frac{1}{2}}\ |\vec{k}_1|$ and centered at $(-2k_1,\ -2l_1)$, since (A.4) can be written as

$$(l_2 + 2l_1)^2 + (k_2 + 2k_1)^2 = 3(k_1^2 + l_1^2). \tag{A.5}$$

Finally, since $\vec{k}_2$ must also satisfy $\omega = \sigma(\vec{k}_2)$, they must also lie on the same slowness curve that passes through $\vec{k}_1$, i.e. the circle of radius $(\gamma^2 - r_e^{-2})^{\frac{1}{2}}$ and centered at $(\gamma,\ 0)$, where $\gamma = -\beta/(2\omega)$ and $\omega = \sigma(\vec{k}_1)$. The geometric locus of wavenumber vectors $\vec{k}_2$ that resonantly interact with the given $\vec{k}_1$, and such that $\omega_1 = \omega_2$, is given by the intersection of these two circles, and therefore, there are at most two (see Fig. 3). The high values of $U_s^{(1)}$ in Figure 1(b) for $\theta_2 \sim 240^\circ$and Λ_2 in the range 200–350 km correspond to resonances with $\omega_1 = \omega_2$, as confirmed by the two vectors $\vec{k}_2$ shown in Figure 3 (wave–1 is the same in Figs. 1 and 3) and by looking at the level curve of annual period for waves–2 (not shown). Note that the more $\vec{k}_1$ is westward, the less possibility there is for an intersection, i.e. for resonance.

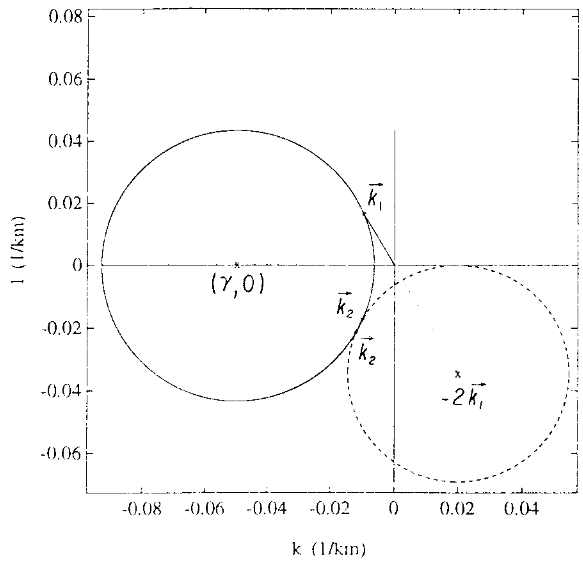

Figure 3. Graphical method to find a resonantly interacting triad of RWs at $O(\varepsilon)$ in the case $\omega_1 = \omega_2 = \omega$ for given wavenumber vector $\vec{k}_1$ of wave–1. First, choose a $\vec{k}_1$, compute the frequency from $\omega = \sigma(\vec{k}_1)$ and draw the slowness curve (solid circle) on which all wavenumbers of this frequency must lie, i.e. the circle of radius $(\gamma^2 - r_e^{-2})^{1/2}$ and centered at $(\gamma,\ 0)$. Second, draw the circle (dashed)of radius $3^{1/2} \mid \vec{k}_1 \mid$ and centered at $(-2k_1,\ -2l_1) = -2\vec{k}_1$. The intersections of these two circles (marked with a small *) give the geometric locus of wavenumber vectors $\vec{k}_2$ that resonantly interact with $\vec{k}_1$ and $\vec{k}_1 + \vec{k}_2$. Reference latitude $= 30°$, vertical mode number of both waves $n = 1$, baroclinic Rossby radius of deformation $= 41$ km and $\vec{k}_1$ chosen from the polar coordinates $\theta_1 = 120°$and $\mid \vec{k}_1 \mid = 2\pi/315\ \mathrm{km}^{-1}$, thus resulting in a period of wave–1 $= 1$ year (as in Fig. 1).

The coordinates of the points of intersection can be found analytically by obtaining first the equation of the radical axis of the two circles (a straight line) and then substituting it into either of the circles' equations. The result is

$$k_{2\pm}^{(res)} = \frac{\gamma - mb \pm [(mb-\gamma)^2 - (1+m^2)(b^2 + r_e^{-2})]^{\frac{1}{2}}}{1+m^2}, \tag{A.6}$$

$$l_{2\pm}^{(res)} = mk_{2\pm}^{(res)} + b, \tag{A.7}$$

where $m = -(k_1 + \gamma/2)/l_1$ and $b = \left(r_e^{-2} - \mid \vec{k}_1 \mid^2\right)/(4l_1)$. Obviously, if the circles do (do not) intersect, the radicand in (A.6) must be positive (negative), and if the circles are tangent, the radicand is zero.

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