Atmósfera (1998) 11, pp. 143-155

On the structure of the stability matrix in the normal mode stability study of zonal incompressible flows on a sphere

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(Manuscript received February 20, 1997; accepted in final form November 24, 1997)

RESUMEN

Se analiza la estructura de la matriz que surge en el análisis de la estabilidad lineal (de modos normales) de flujos zonales no divergentes sobre una esfera. El análisis está basado en el uso de una fórmula de recurrencia, derivada para los coeficientes de la interacción no lineal de triadas. En calidad de una aplicación, se demuestra que un flujo de la forma de un polinomio de Legendre de grado j es estable, tanto exponencialmente como algebraicamente, respecto a todas las perturbaciones de escalas pequeñas cuyos números zonales superan j.

ABSTRACT

A structure of the matrix resulting from the normal mode stability of zonal non-divergent flows on a sphere is analyzed. The analysis is based on using the recurrent formula derived for the nonlinear triad interaction coefficients. As an application, it is shown that a zonal flow of the form of a Legendre polynomial of degree j is exponentially and algebraically stable to all the small-scale perturbations whose zonal wave number is greater than j.

1. Introduction

Zonal incompressible two-dimensional flows have been analyzed in many papers devoted to the stability problem of atmospheric circulation (Kuo, 1949; Baines, 1976; Dikiy, 1976; Tung, 1981; Zeng *et al.*, 1986a, b; Skiba, 1992, 1993). The choice of such idealized flows simplifies the stability study and provides useful insight into the mechanism of the growth of initial perturbations.

The vorticity equation (VE) governing such motions is of particular hydrodynamical and meteorological interest (Silberman, 1954; Adem, 1956; Adem and Lezama, 1960; Ellsaesser, 1966). Notice that analytical methods of the stability study can be used only if the fluid domain and basic flow are simple in structure. In more complicated cases, numerical methods are necessary. Errors of the discretization of the original differential vorticity equation as well as other types of numerical errors which always accompany any numerical analysis pose major problems. One of them is the credibility of the numerical results obtained. Indeed, if we consider an ideal unforced fluid then the vorticity equation possesses an infinite number of invariants of motions, while the discrete vorticity equation breaks almost all of these invariants. Since these invariants represent certain restrictions on the behavior of VE solutions and their perturbations, it is clear that numerical and analytical stability results can diverge considerably. Therefore it is always of great importance to analyze the convergence of the numerical stability results (Skiba, 1998). Nevertheless, in this work, we get some exact linear stability results using a numerical approach.

As it is well-known, the normal mode (linear) stability study of a flow is reduced to the solution of a spectral probem for the VE operator linearized about such a flow. Our research is devoted to the analysis of the structure of the stability matrix representing this operator in finite-dimensional subspaces of perturbations on a sphere. A recurrent formula for the nonlinear triad interaction coefficients is derived in Section 2 for the case that one of the spherical harmonics is zonal. The normal mode stability matrix for a basic flow on a sphere is obtained in Section 3, while Section 4 treats a symmetric matrix $K_j^{(m)}$. In Section 5, the structure of the normal mode stability matrix is studied in detail in the case that the basic flow has the form of a Legendre polynomial. The results are used in Section 6 to determine exponentially and algebraically stable perturbations of the Legendre polynomial flows (Propositions I and II).

2. Recurrent formula for $K_{j\alpha\gamma}$

Let $\alpha = (m,\ n)$, and $Q_\alpha(\mu)$ be a normalized associated Legendre function of degree n and zonal number m. We now derive a recurrent formula (with respect to j) for the tensor

$$K_{j\alpha\gamma} = -\frac{1}{2}m_\gamma \int_{-1}^{1} Q_\alpha(\mu)Q_\gamma(\mu)dQ_j(\mu) \tag{1}$$

which is the nonlinear triad interaction coefficient $K_{\beta\alpha\gamma}$ (see, for example, Boer, 1983) in the case when the first spectral number β is zonal: $\beta = (0,\ j)$. We will need this formula later to analyze the matrix structure in the normal mode stability study of the Legendre polynomials. We define by

$$k\alpha + l\gamma = (km_\alpha + lm_\gamma,\ kn_\alpha + ln_\gamma) \tag{2}$$

the linear operation for spherical wave numbers $\alpha = (m_\alpha,\ n_\alpha)$ and $\gamma = (m_\gamma,\ n_\gamma)$ and integers k and l. Let $\varepsilon = (0,\ 1)$. We now substitute $j+1$ instead of j in (1) and use twice the recurrent formula

$$D_n^m Q_n^m(\mu) = \mu Q_{n-1}^m(\mu) - D_{n-1}^m Q_{n-2}^m(\mu), \qquad D_n^m = \left\{\frac{n^2 - m^2}{4n^2 - 1}\right\}^{1/2} \tag{3}$$

(Machenauer, 1977), first for the normalized Legendre polynomial Q_{j+1}, and then for μQ_γ. As a result, we obtain

$$K_{j+1,\,\alpha,\,\gamma} = \frac{1}{D_{j+1}}\Big(\Phi_{j\alpha\gamma} + D_{\gamma+\varepsilon}K_{j,\,\alpha,\,\gamma+\varepsilon} + D_\gamma K_{j,\,\alpha,\,\gamma-\varepsilon} - D_j K_{j-1,\,\alpha,\,\gamma}\Big) \tag{4}$$

where $D_j \equiv D_j^o$, and

$$\Phi_{j\alpha\gamma} = -\frac{1}{2}m_\gamma \int_{-1}^{1} Q_j Q_\alpha Q_\gamma d\mu \tag{5}$$

Since

$$Q_j = \frac{d}{d\mu}\left(\frac{D_{j+1}}{j+1}Q_{j+1} - \frac{D_j}{j}Q_{j-1}\right) \tag{6}$$

formulae (1) and (6) give

$$\Phi_{j\alpha\gamma} = \frac{D_{j+1}}{j+1}K_{j+1,\,\alpha,\,\gamma} - \frac{D_j}{j}K_{j-1,\,\alpha,\,\gamma} \tag{7}$$

Substituting (7) in (4) leads to a recurrent formula for $K_{j\alpha\gamma}$:

$$K_{j+1,\,\alpha,\,\gamma} = \frac{j+1}{jD_{j+1}}\Big(D_{\gamma+\varepsilon}K_{j,\,\alpha,\,\gamma+\varepsilon} + D_\gamma K_{j,\,\alpha,\,\gamma-\varepsilon} - \frac{j+1}{j}D_j K_{j-1,\,\alpha,\,\gamma}\Big) \tag{8}$$

Since $Q_o(\mu) = const$ and $Q_1(\mu) = \sqrt{3}\mu$, (1) gives

$$K_{o\alpha\gamma} = 0. \tag{9}$$

$$K_{1\alpha\gamma} = -\sqrt{3}m_\gamma \delta_{\alpha\gamma} \tag{10}$$

where $\delta_{\alpha\gamma} = \delta_{m_\alpha,\,m_\gamma}\delta_{n_\alpha,\,n_\gamma}$ is the product of the Kronecker deltas. Then for $j \geq 2$, coefficients $K_{j\alpha\gamma}$ are given by (8). For example:

$$K_{2\alpha\gamma} = -3\sqrt{5}m_\gamma(D_{\gamma+\varepsilon}\delta_{\alpha,\gamma+\varepsilon} + D_\gamma \delta_{\alpha,\gamma-\varepsilon}) \tag{11}$$

and

$$K_{3,\alpha,\gamma} = -\frac{15\sqrt{7}}{2} m_\gamma \Big\{ D_\alpha D_{\gamma+\varepsilon} \delta_{\alpha,\gamma+2\varepsilon}$$

$$+ D_{\alpha+\varepsilon} D_\gamma \delta_{\alpha,\gamma-2\varepsilon} + \left(D^2_{\gamma+\varepsilon} + D^2_\gamma - \frac{1}{5} \right) \delta_{\alpha,\gamma} \Big\}. \tag{12}$$

Note that $K_{2\alpha\gamma}$ and $K_{3\alpha\gamma}$ were obtained in Boer (1983) by direct calculations. It follows from (1) and (5) that for each fixed j, both $K_{j\alpha\gamma}$ and $\Phi_{j\alpha\gamma}$ are symmetric with respect to α and γ.

3. The stability matrix

Let $\tilde{\psi}(\lambda, \mu)$ be a basic solution of the vorticity equation for a viscous incompressible fluid subjected on the unit sphere S to a forcing and dissipation. Then the evolution of an infinitesimal perturbation $\psi(t, \lambda, \mu)$ of $\tilde{\psi}(\lambda, \mu)$ is governed by

$$\frac{\partial}{\partial t} \varsigma = L\, \varsigma \tag{13}$$

where

$$L\, \varsigma = J(\tilde{\Omega}, \Delta^{-1}\varsigma) - J(\tilde{\psi}, \varsigma) - [\sigma + \nu \Lambda^{2S}]\varsigma \tag{14}$$

is the linear operator (Skiba, 1994a, b), $\varsigma = \Delta\psi$ is the perturbation vorticity,

$$\tilde{\Omega}(\lambda, \mu) = \tilde{\varsigma}(\lambda, \mu) + 2\mu, \qquad \varsigma(\lambda, \mu) = \Delta\tilde{\psi}(\lambda, \mu) \tag{15}$$

σ is the linear drag coefficient, ν is the diffusion coefficient, and the operator Λ^S is defined for any real $S \geq 0$ and any function

$$\psi(\lambda, \mu) = \sum_{\alpha(1)}^{\infty} \psi_\alpha Y_\alpha \equiv \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda, \mu)$$

as

$$\Lambda^S \psi = \sum_{\alpha(1)}^{\infty} \chi_\alpha^{S/2} \psi_\alpha Y_\alpha \equiv \sum_{n=1}^{\infty} \chi_n^{S/2} \sum_{m=-n}^{n} \psi_n^m Y_n^m(\lambda, \mu) \tag{16}$$

where $\alpha = (m, n)$, $Y_\alpha(\lambda, \mu) \equiv Y_n^m(\lambda, \mu)$ is the spherical harmonic, $\psi_\alpha \equiv \psi_n^m$ is the Fourier coefficient, and

$$\chi_\alpha \equiv \chi_n = n(n+1) \tag{17}$$

We use here the notations by Baer and Platzman (1961) and Platzman (1962). The operator Λ is interpreted as the square root of the positive Laplace operator on a sphere: $\Lambda^2 = -\Delta$ (Skiba, 1989, 1994b, 1997).

Assume that the basic solution $\tilde{\psi}$ and perturbation ψ belong to subspaces $\mathbf{P}^M$ and $\mathbf{P}^N$ of spherical polynomials of degrees M and N, respectively:

$$\tilde{\psi} = \sum_{\beta(1)}^{M} \tilde{\psi}_\beta Y_\beta, \qquad \tilde{\Omega} = \sum_{\beta(1)}^{M} \tilde{\Omega}_\beta Y_\beta \tag{18}$$

$$\psi = \Delta^{-1}\varsigma = -\sum_{\alpha(1)}^{N} \chi_\alpha^{-1} \varsigma_\alpha Y_\alpha, \qquad \varsigma = \sum_{\alpha(1)}^{N} \varsigma_\alpha Y_\alpha \tag{19}$$

We now consider the restriction of the operator L to subspace $\mathbf{P}^N$. Substituting (18) and (19) in (13), and taking the inner product of the equation obtained with a hamonic $Y_\alpha(n_\alpha \leq N)$, lead to

$$\frac{d}{dt}\varsigma_\alpha =< L\varsigma, Y_\alpha >\equiv \int_{\mathbf{S}} (L\varsigma)\overline{Y}_\alpha dS = \sum_{\gamma(1)}^{N} L_{\alpha\gamma}\, \varsigma_\gamma \tag{20}$$

where

$$L_{\alpha\gamma} =< LY_\gamma, Y_\alpha > \tag{21}$$

are the (α, γ)-element of the matrix L representing the operator (14) in the subspace $\mathbf{P}^N$ (Skiba, 1989). Thus in $\mathbf{P}^N$, problem (13) is reduced to

$$\frac{d}{dt}\vec{\varsigma} = L\vec{\varsigma} \tag{22}$$

where vector $\vec{\varsigma}$ of $\mathbf{P}^N$ has components $\varsigma_n^m (n = 1, 2, \ldots, N; \mid m \mid\leq n)$.

Substituting (14) in (21) gives

$$L_{\alpha\gamma} =< J(\tilde{\Omega}, \Delta^{-1}Y_\gamma), Y_\alpha >$$

$$- < J(\tilde{\psi}, Y_\gamma),\ Y_\alpha > - < (\sigma + \nu\Lambda^{2S})Y_\gamma, Y_\alpha >$$

$$= - < J(\tilde{\psi} + \chi_\gamma^{-1}\tilde{\Omega}, Y_\gamma), Y_\alpha > -(\sigma + \nu\chi_\gamma^S)\delta_{\alpha\gamma} \tag{23}$$

Due to (18) and (15),

$$\tilde{\psi} + \chi_\gamma^{-1}\tilde{\Omega} = 2\mu\chi_\gamma^{-1} + \sum_{\beta(1)}^{M} (\chi_\gamma^{-1} - \chi_\beta^{-1})\tilde{\varsigma}_\beta Y_\beta \tag{24}$$

Taking into account that

$$< J(\mu, Y_\gamma), Y_\alpha >= - < \frac{\partial}{\partial \lambda} Y_\gamma, Y_\alpha >= -i m_\gamma \delta_{\alpha\gamma} \tag{25}$$

we obtain

$$L_{\alpha\gamma} = \sum_{\beta(1)}^{M} B_{\beta\alpha\gamma} \tilde{\varsigma}_\beta + D_{\alpha\gamma} \tag{26}$$

where

$$B_{\beta\alpha\gamma} = \left(\chi_\beta^{-1} - \chi_\gamma^{-1}\right) < J\left(Y_\beta, Y_\gamma\right), Y_\alpha > \tag{27}$$

is symmetric over β and γ, the scalar product

$$< J(Y_\beta, Y_\gamma), Y_\alpha >= \begin{cases} o & \text{if } m_\beta + m_\gamma \neq m_\alpha \\ \mathrm{i}4\pi K_{\beta\alpha\gamma} & \text{if } m_\beta + m_\gamma = m_\alpha \end{cases} \tag{28}$$

characterizes the resonance interaction of the three spherical harmonics (Silberman, 1954), and

$$D_{\alpha\gamma} = \left\{-(\sigma + \nu\chi_\gamma^S) + \mathrm{i}2m_\gamma\chi_\gamma^{-1}\right\} \delta_{\alpha\gamma} \tag{29}$$

is the complex element of a diagonal matrix D. Note that for $S = 2$, this matrix was first given by Simmons *et al.* (1983). By (29), the linear drag, viscosity and Earth's rotation contribute only in the diagonal elements $D_{\alpha\alpha}$ of matrix L, besides, the real and imaginary parts of $D_{\alpha\alpha}$ represent the dissipation and Earth's rotation, respectively.

Also note that the use of orthonormal basis of the spherical harmonics leads to the complexification of the real space of solutions on a sphere. Therefore the elements $L_{\alpha\gamma}$ of matric L obey

$$L_{\overline{\alpha}\,\overline{\gamma}} = (-1)^{m_\alpha + m_\gamma} \overline{L_{\alpha\gamma}} \tag{30}$$

Indeed,

$$L_{\overline{\alpha}\,\overline{\gamma}} =< LY_{\overline{\gamma}}, Y_{\overline{\alpha}} >= (-1)^{m_\alpha + m_\gamma} < L\overline{Y_\gamma}, \overline{Y_\alpha} >$$

$$= (-1)^{m_\alpha + m_\gamma} < \overline{LY_\gamma, Y_\alpha} >= (-1)^{m_\alpha + m_\gamma} \overline{L_{\alpha\gamma}} \tag{31}$$

4. Matrix $K_j^{(m)}$

For integers j and $m (1 \leq j \leq M,\ 1 \leq |\, m \,| \leq N)$, we introduce a symmetric matrix $\mathbf{K}_j^{(m)}$ with the entries

$$a_{kl} = -\frac{m}{2} \int_{-1}^{1} Q_k^m(\mu) Q_l^m(\mu) dQ_j(\mu) \tag{32}$$

where $Q_k^m(\mu)$ is the normalized associated Legendre function, $Q_j(\mu) \equiv Q_j^o(\mu)$, and indices k and l are changed from $\mid m \mid$ to N. The elements of the matrix $\mathbf{K}_j^{(m)}$ are obviously the nonlinear triad interaction coefficients (1) under the conditions that $m_\alpha = m_\gamma = m$.
Note that

$$\mathbf{K}_j^{(-m)} = -\mathbf{K}_j^{(m)} \tag{33}$$

Due to the recurrent formula (8) and the selection rules (Silberman, 1954), it is easily obtained that for any j and $m (1 \le j \le M,\ 1 \le \mid m \mid \le N)$, matrix $\mathbf{K}_j^{(m)}$ is a real symmetric banded matrix (i.e., $a_{kl} = 0$ if $p < k - l$ and $l - k > q$ for some non-negative integers p, q). For even j, the band width of $\mathbf{K}_j^{(m)}$ is $2j - 1$, besides, inside the band, $a_{kl} = 0$ if $k + l$ is even, i.e., the diagonals with zero and nonzero elements alternate (Fig. 1a). For odd j, the matrix $\mathbf{K}_j^{(m)}$ is block-diagonal (Fig. 1b):

$$\mathbf{K}_j^{(m)} = Z_j^{(m)} \oplus R_j^{(m)} \tag{34}$$

where $Z_j^{(m)}$ and $R_j^{(m)}$ are symmetric banded matrices whose band width is equal to j. The elements of $Z_j^{(m)}$ (or $R_j^{(m)}$) are the entries (32) with odd (even) numbers k and l. Both $Z_j^{(m)}$ and $R_j^{(m)}$ have no zero diagonals inside their bands (Fig. 1c).

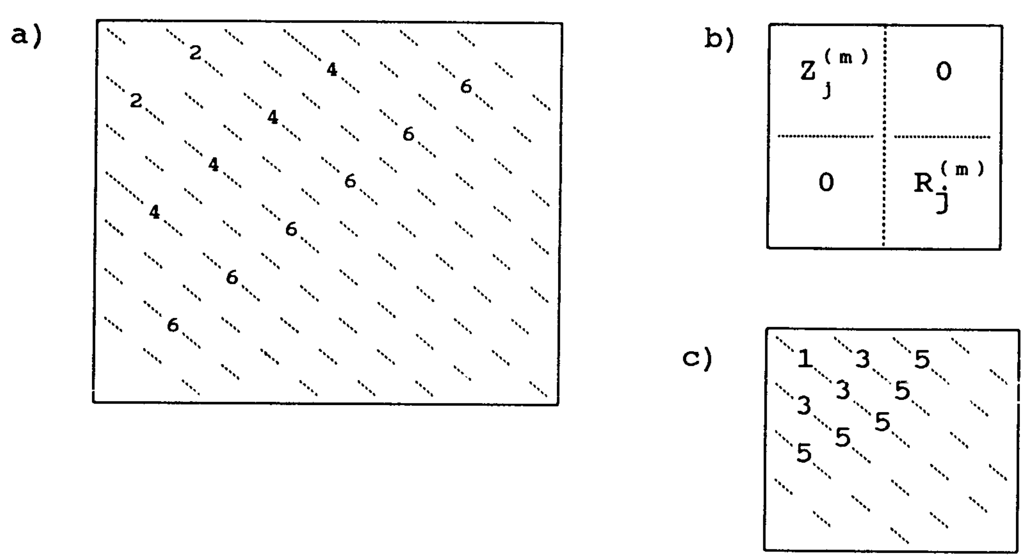

Fig. 1. The structure of the matrix $\mathbf{K}_j^{(m)}$

a. The banded structure of the matrix $\mathbf{K}_j^{(m)}$ for $j = 2, 4, 6$ and a fixed m. Nonzero diagonals of the matrices $\mathbf{K}_2^{(m)}$, $\mathbf{K}_4^{(m)}$ and $\mathbf{K}_6^{(m)}$ are marked by the numbers 2, 4 and 6, respectively.

b. Block diagonal structure of the matrix $\mathbf{K}_j^{(m)} = Z_j^{(m)} \oplus R_j^{(m)}$ for odd j.

c. The banded structure of the matrix $Z_j^{(m)}$ for $j = 1, 3, 5$ and a fixed m. Nonzero diagonals of the matrices $Z_1^{(m)}$, $Z_3^{(m)}$ and $Z_5^{(m)}$ are marked by the numbers 1, 3 and 5, respectively. The matrices $R_j^{(m)}$ have the same structure.

Let $2 \leq | m | \leq N$. Then any matrix $\mathbf{K}_j^{(m)}$ with even j as well as any of the matrices $Z_j^{(m)}$, $R_j^{(m)}$ and $R_j^{(1)}$ with odd j are irreducible. It follows from the fact that their elements $a_{k,k-1}$ and $a_{k,k+1}$ are nonzero for any k according to the selection rules, and hence the directed graph of any of these matrices is strongly connected (Lancaster, 1969). For $| m |= 1$, the matrices $\mathbf{K}_{2n}^{(1)}$ and $Z_{2n-1}^{(1)}$ are reducible according to one of the selection rules (Fig. 2a, b). However if the stability analysis is restricted by considering only such perturbations ς of the set $\mathbf{I}_1$ that are orthogonal to the spherical harmonic $Y_1^1(\lambda, \mu)$ then matrices $\mathbf{K}_{2n}^{(1)}$ and $Z_{2n-1}^{(1)}$ become irreducible. The truth of such a restriction follows from the fact that the Fourier coefficients $\varsigma_1^o(t)$ and $\varsigma_1^1(t)$ of the vorticity perturbation $\varsigma(t)$ are time invariant. Indeed, $\varsigma_1^o(t) = \text{const}$ due to the conservation of the angular momentum, while $\varsigma_1^1(t) = \text{const}$ due to the conservation of both ς_1^o and the part of the solution kinetic energy concentrated in the subspace of homogeneous spherical polynomials of the degree one (Skiba, 1989).

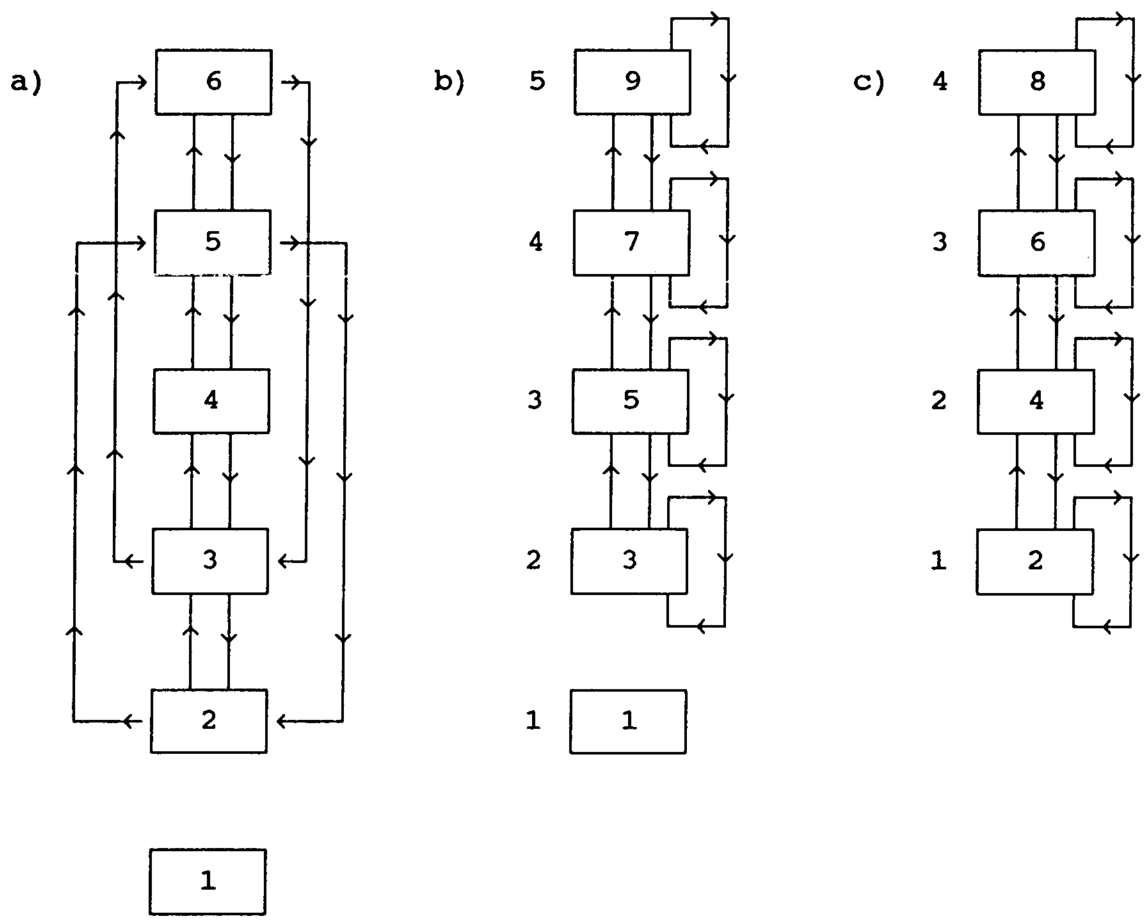

Fig. 2. Directed graph of the matrix $\mathbf{K}_4^{(1)}$ for $N = 6$ (a), and matrices $Z_3^{(1)}$ (b) and $R_3^{(1)}$ (c) for $N = 9$.

Thus a further decomposition of any of the matrices $\mathbf{K}_j^{(m)}$ into diagonal blocks is impossible. In particular, $\mathbf{K}_j^{(0)}$ is zero for any $j \geq 1$. Also for each non-zero $m (1 \leq m \leq N)$, $\mathbf{K}_1^{(m)}$ is the scalar matrix with the elements $-\sqrt{3}m$ on the principal diagonal, and any of $\mathbf{K}_2^{(m)}$, $Z_3^{(m)}$ and $R_3^{(m)}$ is three-diagonal Jacobi matrix, since $a_{k,k+1}a_{k+1,k}$ is positive for any k (Voevodin and Kuznetzov, 1984).

5. Stability matrix of a Legendre-polynomial flow

Since a zonal flow can be represented as Fourier's series of the Legendre polynomials, it is of special interest to analyze the stability of the flow which has the form of a Legendre polynomial:

$$\tilde{\psi}(\mu) = \tilde{\psi}P_\beta(\mu) = \tilde{\psi}P_j(\mu)$$

$$\tilde{\varsigma}(\mu) = aP_j(\mu) \tag{35}$$

where $\beta = (0,\ j)$, $j \geq 2$, and $\tilde{\psi}$ and $a = -\chi_j\tilde{\psi} = -j(j+1)\tilde{\psi}$ are the amplitudes. Then there is no summation in (26), and the elements of the matrix L are

$$L_{\alpha\gamma} = \tilde{\varsigma}_\beta B_{\beta\alpha\gamma} + D_{\alpha\gamma} = a(\chi_j^{-1} - \chi_\gamma^{-1}) < J(P_j, Y_\gamma), Y_\alpha > + D_{\alpha\gamma} \tag{36}$$

Due to (35) and selection rules, $B_{\beta\alpha\gamma} = 0$ if $m_\gamma \neq m_\alpha$, or $m_\gamma = 0$. Taking account of (29) we obtain that L is a block diagonal matrix:

$$L = \bigoplus_{m=-N}^{N} L^{(m)}, \quad L^{(m)} = \mathrm{i}V^{(m)} - D^{(m)} \tag{37}$$

where i is the imaginary unit, and

$$V^{(m)} = aK_j^{(m)}C^{(m)} + T^{(m)} \tag{38}$$

Each $L^{(m)}$ is a restriction of L to the subspace $\mathbf{I}_m^N = \mathbf{I}_m \cap \mathbf{P}_o^N$ where $\mathbf{I}_m$ is the span of the spherical harmonics $Y_n^m(\lambda, \mu)$ with $n \geq \mid m \mid$. In (38), $K_j^{(m)}$ is the symmetric matrix (32), and

$$C^{(m)} = diag\{c_k\}, \qquad T^{(m)} = diag\{T_k\}, \qquad D^{(m)} = diag\{d_k\} \tag{39}$$

are diagonal matrices of the same order as $K_j^{(m)}$, besides, according to (29) and (36) their elements are

$$c_k = 4\pi(\chi_j^{-1} - \chi_k^{-1}), \qquad T_k = 2m\chi_k^{-1}, \qquad d_k = \sigma + \nu\chi_k^S$$

$$(k = \mid m \mid, \mid m \mid +1, \ldots, N) \tag{40}$$

where χ_k is given by (17). For odd j, each matrix $L^{(m)}(1 \leq \mid m \mid \leq N)$ is block diagonal due to (34):

$$L^{(m)} = L_o^{(m)} \oplus L_E^{(m)} = \mathrm{i}\{a[Z_j^{(m)} \oplus R_j^{(m)}]C^{(m)} + T^{(m)}\} + D^{(m)} \tag{41}$$

In other words, $L_o^{(m)}$ and $L_E^{(m)}$ are the restrictions of matrix L to the subspaces $\mathbf{O}_m^N = \mathbf{O}_m \cap \mathbf{P}^N$ and $\mathbf{E}_m^N = \mathbf{E}_m \cap \mathbf{P}^N$, respectively, where $\mathbf{O}_m$ is the span of all $Y_n^m(\lambda, \mu)$ of $\mathbf{I}_m$ whose degree n is odd, and $\mathbf{E}_m = \mathbf{I}_m \ominus \mathbf{O}_m$ is the orthogonal complement of $\mathbf{O}_m$ to $\mathbf{I}_m$.

6. Stable invariant manifolds of a Legendre-polynomial flow

We now find stable invariant sets of infinitesimal perturbations of the Legendre polynomial flow (35) using the structure (37) and (38) of the stability matrix (for more comprehensive results see Skiba and Adem, 1998).

Proposition I. ***Let $\sigma, \nu \geq 0$ in (40), and j be a degree of the Legendre polynomial flow (35). If $m = 0$ or $| m |> j$ then for each truncation number N of series (19), the subspace $\mathbf{I}_m^N = \mathbf{I}_m \cap \mathbf{P}^N$ of dimension $r = N - | m | + 1$ belongs to the stable manifold of the flow (35).***

Proof. **1. Let first $m = 0$. Then $V^{(o)} = 0$ and $L^{(o)} = -D^{(o)}$, and hence for each j, the set $\mathbf{I}_o^N = \mathbf{I}_o \cap \mathbf{P}^N$ belongs to the stable manifold of the basic flow (35). The decay rate of each zonally symmetric perturbation is determined by the corresponding diagonal element of the matrix $D^{(o)}$. In an inviscid fluid ($D^{(o)} = 0$), the set $\mathbf{I}_o^N$ belongs to the kernel of matrix L.**

2. Let $| m |> j$. For the sake of simplicity we will omit the fixed indices m and j, i.e., denote the matrices $V^{(m)}$, $K_j^{(m)}$, $C^{(m)}$, $T^{(m)}$ and $D^{(m)}$ in the subspace $\mathbf{I}_m^N$ by the symbols V, K, C, T and D, respectively. Then matrices C and $R = CD$ are positive definite, and

$$X = CV = aCKC + CT \tag{42}$$

is a real symmetric matrix. In each subspace $\mathbf{I}_m^N$, the problem (22) has the form

$$\frac{d}{dt}\vec{\varsigma} = \mathbf{L}^{(m)}\vec{\varsigma} = (\mathrm{i}V - D)\vec{\varsigma} \tag{43}$$

or

$$\frac{d}{dt}(C\vec{\varsigma}) = \mathrm{i}X\vec{\varsigma} - R\vec{\varsigma} \tag{44}$$

Consider the equation

$$\left(\frac{d}{dt}\vec{\varsigma}^{\,*}\right) C = -\mathrm{i}\vec{\varsigma}^{\,*}X - \vec{\varsigma}^{\,*}R \tag{45}$$

adjoint to (44). Multiplying (44) to the left by vector $\vec{\varsigma}^{\,*}$, and (45) to the right by vector $\vec{\varsigma}$, and

summing the results give

$$\frac{d}{dt}(\vec{\varsigma}^{\,*}C\vec{\varsigma}) = -2\vec{\varsigma}^{\,*}R\vec{\varsigma} \leq 0 \tag{46}$$

In the case of an ideal fluid, the product $\vec{\varsigma}^{\,*}C\vec{\varsigma}$ is conserved. It follows from here that the Legendre polynomial flow (35) is stable to infinitesimal perturbations $\vec{\varsigma}$ of the subspace $\mathbf{I}_m^N$ in the norm (Liapunov function) $\| \vec{\varsigma} \|_c = (\vec{\varsigma}^{\,*}C\vec{\varsigma})^{1/2}$. The assertion is proved.

It follows from Proposition I that for an ideal fluid ($\sigma = 0,\ \nu = 0$), infinitesimal perturbations of (35) of $\mathbf{I}_m^N$ ($| m | > j$) can not grow not only exponentially but also algebraically. The last fact follows immidiately from the next assertion.

Proposition II. *Let $\sigma = \nu = 0$ and let N and j be natural. Then for each zonal number m satisfying $| m | > j$, the matrix $L^{(m)}$ has the simple structure.*

Proof. Indeed, in this case each $(r \times r)$-matrix $L^{(m)}$ is defined as $L = \mathrm{i}V - \sigma E$ where index m is omitted and $r = N - | m | + 1$. Therefore it is sufficient to show that the eigenvalues of the spectral problem

$$V\vec{\varphi}_n = \omega_n\vec{\varphi}_n \tag{47}$$

are linearly independent. Consider first the case $| m | > j$. Then the matrix $C \equiv C^{(m)}$ is positive, and (47) is equivalent to the spectral problem

$$X\vec{\varphi}_n - \omega_n C\vec{\varphi}_n \tag{48}$$

with a symmetric matrix (42). Therefore, by theorem (15.3.4) from Parlett (1980), spectral problem (48), and hence (47), has in the interval $[-\| V \|, \| V \|]$ exactly r real eigenvalues $\omega_1 \ldots, \omega_r$ whose eigenvectors $\vec{\varphi}_1, \ldots, \vec{\varphi}_r$ are C-orthogonal,

$$\vec{\varphi}_k^{\,*}C\vec{\varphi}_n = \delta_{kn} \tag{49}$$

and hence, linearly independent. Here $\| V \|$ is the spectral norm of matrix (38). The assertion is proved.

7. Final conclusion

A recurrent formula is derived for the nonlinear triad interaction coefficients in the case when one of the wave numbers is zonal. Using this formula, a structure of the matrix resulting from the normal model stability of zonal non-divergent flows on a sphere is analyzed. A knowledge of the matrix structure is used to show that in an incompressible fluid on a sphere (both viscous and ideal), a zonal flow of the form of a Legendre polynomial of degree j is exponentially and algebraically stable to all the small-scale perturbations whose zonal wave number m is larger than j (Propositions I and II).

Acknowledgements

We thank to anonymous referees for their useful comments and criticism regarding the manuscript. We are also grateful to Mrs. Thelma del Cid for editing the text, and Mrs. M. E. Grijalva for the final preparation of the manuscript.

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