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On the stable part of the Lorenz attractor

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RESUMEN

Para un valor subcrítico del número de Rayleigh el atractor de Lorenz tiene un estado de permanencia inestable y dos estables. El estado asintótico de una integración de largo tiempo será uno de los dos estados permanentes. Un estado de permanencia estable será el estado asintótico si la posición inicial se encuentra en la cercanía inmediata del estado de permanencia. Sin embargo, en la región intermedia entre dos estados estables del estado asintótico puede hallarse sólo por una integración numérica. Pequeños cambios en el estado inicial pueden causar un cambio en el estado asintótico de un estado de permanencia estable al otro.

El estado asintótico final es sensible no sólo a los cambios pequeños en el estado inicial, sino también a pequeños cambios en el valor del número de Rayleigh. Este comportamiento indica que al menos la región fronteriza puede ser de naturaleza fractal.

En vista del comportamiento de las integraciones temporales fue formulado un modelo estocástico, incluyendo sólo estadísticas de segundo orden. Este modelo puede usarse para investigar los estados asintóticos en función de la magnitud de la incertidumbre en el estado inicial. Se darán varios ejemplos.

Se compara el modelo estocástico con una solución exacta de los estados de permanencia dadas las posiciones iniciales y las variancias.

ABSTRACT

For a subcritical value of the Rayleigh number the Lorenz attractor has one unstable and two stable stationary states. The asymptotic state of a long time integration may one of the two steady states. A stable steady state will be the asymptotic state if the starting position is in the immediate neighbourhood of the steady state. However, in the region midway between the two stable states the asymptotic state can be found only by a numerical integration. Small changes in the initial state may cause a change in the asymptotic state from one stable steady state to the other.

The final asymptotic state is sensitive not only to the small changes in the initial state, but also to small changes in the value of the Rayleigh number. This behavior indicates that at least the border region may be of a fractal nature.

In view of the behavior of the time integrations a stochastic model, including only second order statistics, was formulated. This model can be used to investigate the asymptotic states as a function of the magnitude of the uncertainty in the initial state. Several examples will be given.

The stochastic model is compared with an exact solution for the steady states given the initial position and the initial variances.

1. Introduction

The nature of the Lorenz attractor (Lorenz, 1963) has been described in earlier investigations too numerous mention. The main result is that a critical value of the Reynolds number ($r_c = 24.74$) determines a change from one unstable and two stable to three unstable steady states. The latter case has been considered the most interesting and has been used to demonstrate the concept of limited predictability being identical to a chaotic behavior, while the former case has been considered of much smaller interest, because the final state of any time integration may be one or the other of the two stable steady states. One may, however, ask if it is possible to determine in advance of the integration which one of the two possible asymptotic states will be selected. Needless to say, such a prediction is straightforward if the initial condition is sufficiently close to a particular steady state. On the other hand, if the initial state does not satisfy this criterion, it is much more difficult to answer the above question.

For the case of three unstable steady states it is straightforward to demonstrate that the limited predictability may be caused also by an uncertainty in the value of the Rayleigh number assuming that the two integrations start from identical initial states. The same situation will be investigated by numerical integrations for the case in which two stable steady states are present.

The detailed nature of the Lorenz attractor is unknown. We are thus restricted to numerical integrations. Any such integration is influenced by the selected scheme for the time integration. If it turns out that the attractor in a certain region is very complicated in the sense that a very small change in the initial state or in the Rayleigh number results in different asymptotic states, it is conceivable that a change in the time integration scheme may lead to a different result. This question will be investigated, but whatever the answer it should be noted that any time difference scheme may be improved to assure higher accuracy, and one will thus never be able to provide a definite answer.

In view of the sensitivity to small changes in the initial state it may be an advantage to incorporate the uncertainty in the equations themselves. Such a procedure has been used by Epstein (1969), Fleming (1971) and Epstein and Pitcher (1972). We shall adopt this procedure for the Lorenz equations and derive the proper set of equations including, however, only second order statistics. This stochastic system will be analysed and used to illustrate different behaviors for small and large uncertainties in the initial state. The restriction to second order statistics is probably a reasonable approximation. On the other hand, the general procedure will always require an approximation in order to close the system, and, just as with finite differences, we do not know a preferable closure approximation.

2. Experiments with the Lorenz model

The equations for the Lorenz model are:

$$\begin{aligned} \frac{dx}{dt} &= s(x-y) \\ \frac{dy}{dt} &= -xz + rx - y \\ \frac{dz}{dt} &= xy - bz \end{aligned} \tag{2.1}$$

in which standard notation has been used except that the original σ has been replaced by s. For s and b we shall in the whole investigation apply the usual values: s = 10.0 and b = 8/3, while r, the Rayleigh number, will be varied. The limited predictability for small changes in the Rayleigh number for $r > r_c$ may be illustrated by making two integrations from identical initial conditions, but with slightly different values of the Rayleigh number. For this purpose we select r = 28.0 and r_1 = 28.0000001 and x_o = 0.0, y_o = 0.1 and z_o = 0.0 in both cases. The difference between the two solutions will be measured by the RMS-difference between them, i.e.

$$d = [(x-u)^2 + (y-v)^2 + (z-w)^2]^{1/2} \tag{2.2}$$

in which x, y, z and u, v, w are the dependent variables in the two integrations. Figure 1 shows that the difference becomes large when the time variable is about 0.6 whereafter it decreases and indicates an undamped oscillation. On the other hand, when we change to the subcritical values of r = 23.0 and r_1 = 23.0000001 and perform the integration we find the result given in Figure 2. In this case it is seen that the RMS-difference approaches a constant value. It is easy to check that this value (21.66) is identical to the distance between the two steady states. They are in general x = y = ± $(b(r-1))^{½}$ and z = r-1 as determined from the steady state equations. Consequently, we have found an example in which the two integrations finish in separate stable steady states.

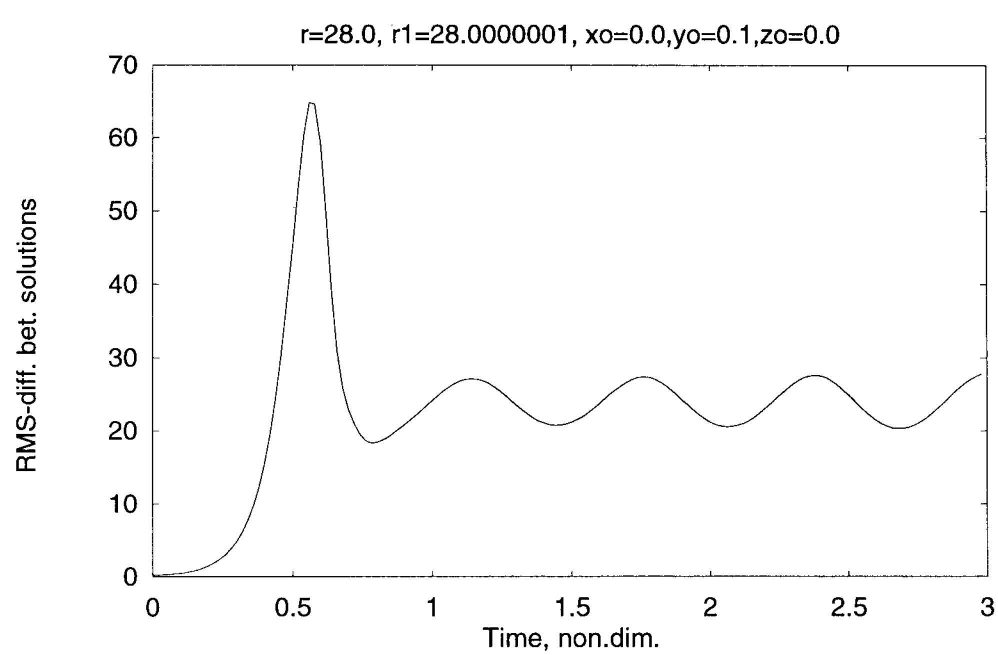

Fig. 1. The RMS-difference between two solutions with r = 28.0 and r = 28.0000001 and with common initial state (0.0,0.1,0.0).

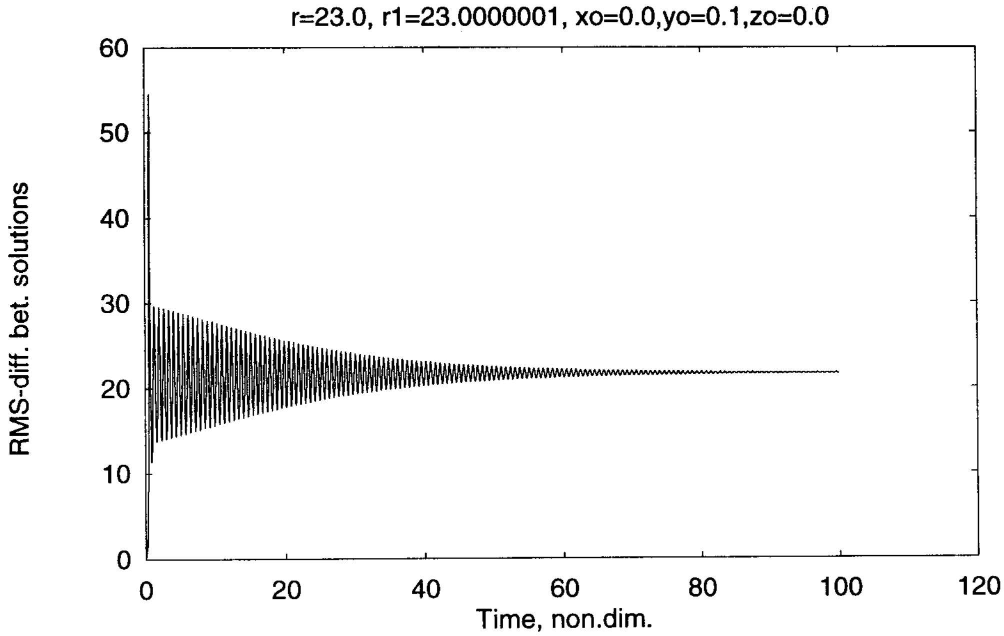

Fig. 2. The RMS-difference between two solutions with r = 23.0 and r = 23.0000001 and with common initial state (0.0,0.1,0.0).

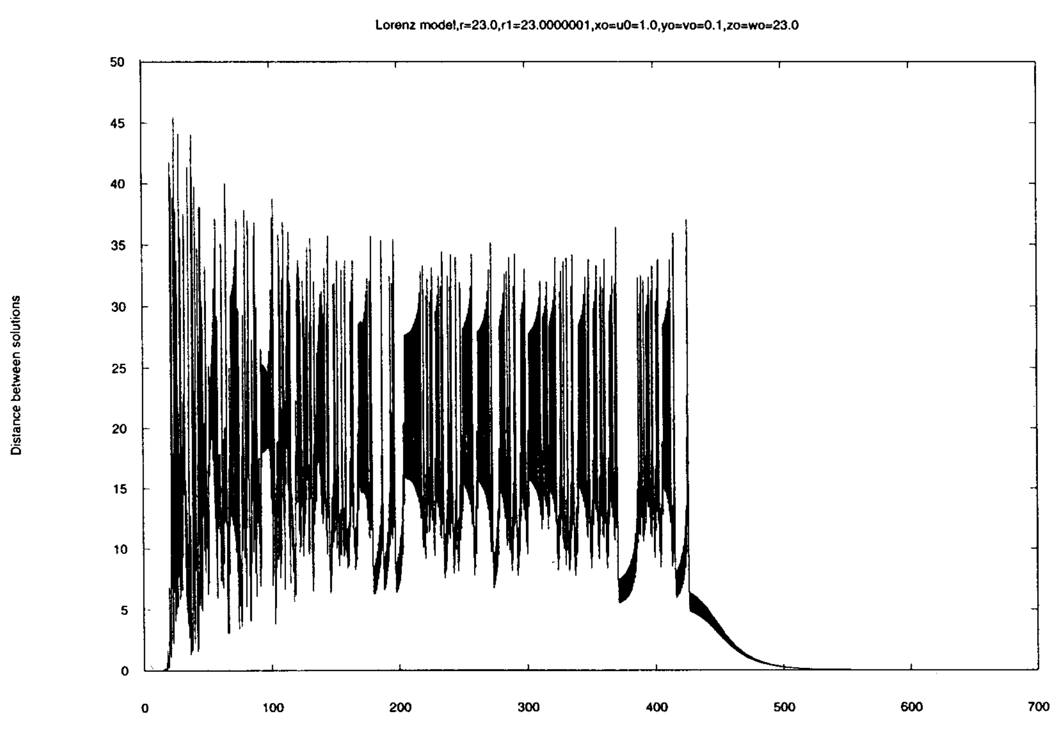

Fig. 3a. The RMS-difference between two solutions with r = 23.0 and r = 23.0000001 and with common initial state (1.0,1.0,23.0).

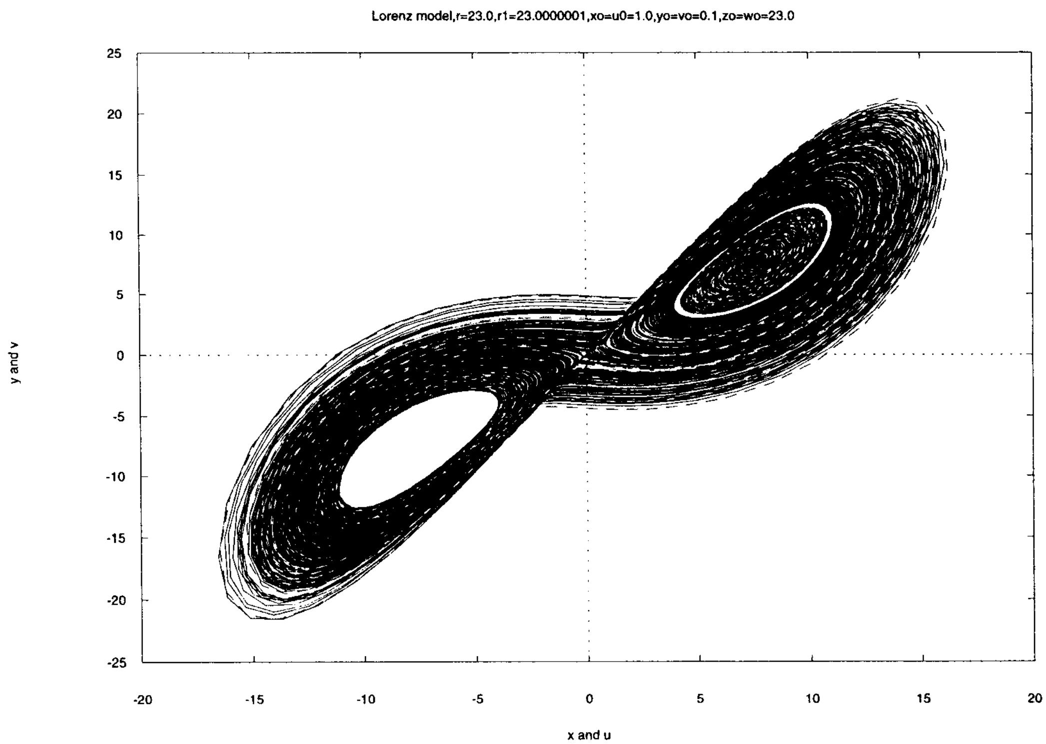

Fig. 3b. The two trajectories for the cases treated in Fig. 3a.

It is possible to find examples in which the two integrations lead to the same steady state. Figure 3a shows the RMS-difference for a case with $r = 23.0$, $r_1 = 23.0000001$ and the common initial state $x_o = 1.0$, $y_o = 0.1$ and $z_o = 23.0$. This rather long integration shows that d vanishes after an integration of 600 time-units. Figure 3b shows the projection on the (x, y) or (u, v) plane indicating that the two integrations both finish in the steady state in the first quadrant.

To illustrate the complex behavior we show the results of three integrations in which $r = 23.0$ and $r_1 = 23.0000001$, $x_o = u_o = 1.0$ and $z_o = w_o = 23.0$ are the same in all three cases. Figure 4a shows $x = x(t)$ and $u = u(t)$ for $y_o = v_o = -0.6$. It is evident that the two parallel integrations lead to the same steady state characterized by positive values of x and y. It is, however, also clear that they arrive in the same steady state at quite different times. In Figure 4b we have changed to an initial state of $y_o = v_o = -0.5$. In this case the two integrations arrive in different steady states and at greatly different times. Finally, in the third case we have used $y_o = v_o = -0.3$. As indicated by Figure 4c the two integrations finish at about the same time in the same steady state, but this time in the state with negative values of x and y.

All integrations mentioned so far have used the Heun time differencing scheme. It is the lowest order Runge-Kutta scheme with second order accuracy. Since the integrations show that the final steady state is sensitive to both small variations in the initial state and in the Rayleigh number, it is likely that the asymptotic state is also sensitive to the selected time differencing scheme. The integrations were first repeated using a fourth order Runge-Kutta scheme with a fixed, but very small time (dt = 0.0001). The same results with respect to the asymptotic state as those described above were found.

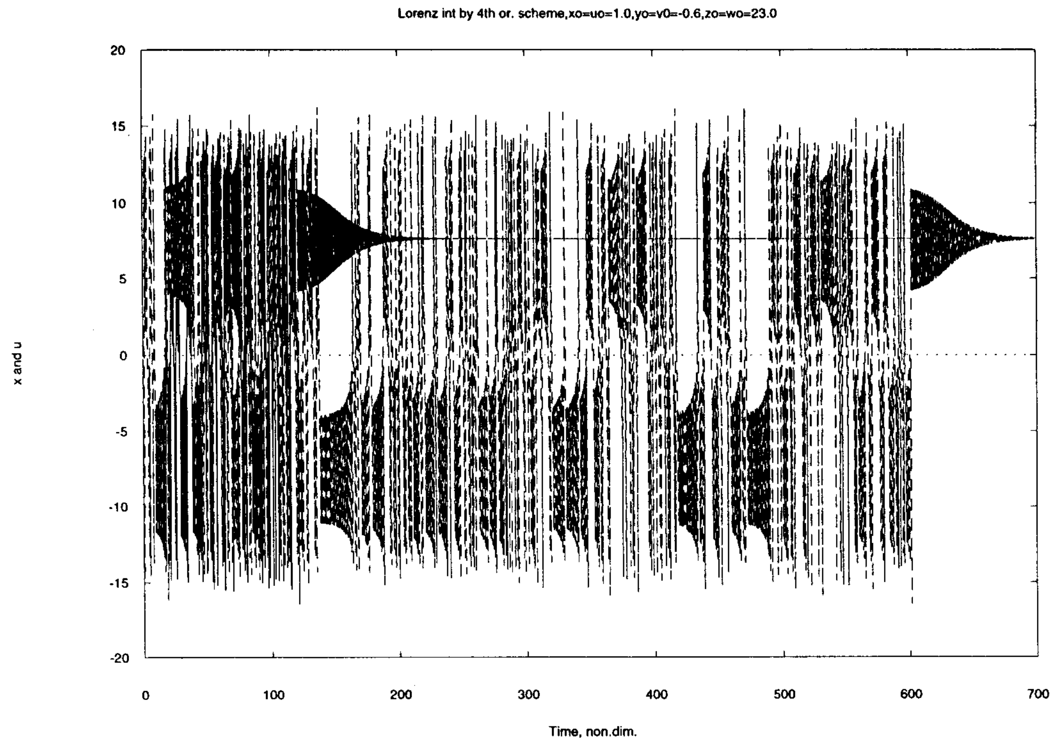

Fig. 4a. The time dependence, x = x(t) and u = u(t), of two solutions with r = 23.0 and r = 23.0000001 and the common initial state (1.0,-0.6,23.0).

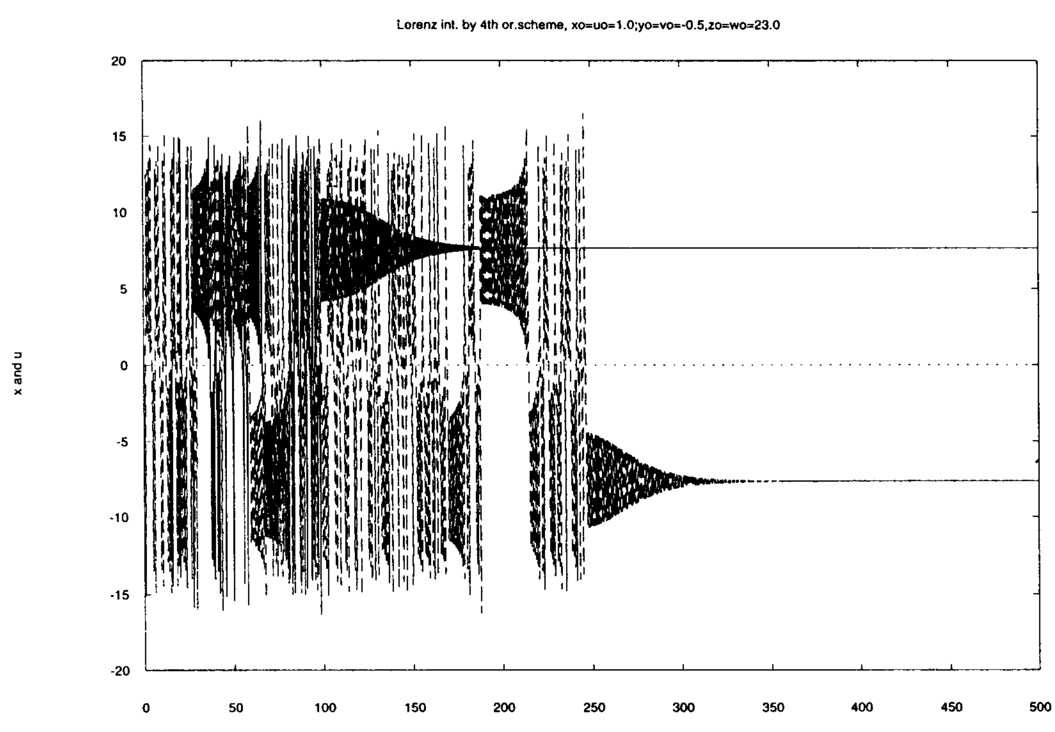

Fig. 4b. As Fig. 4a except that the common initial state is (1.0,-0.5, 23.0).

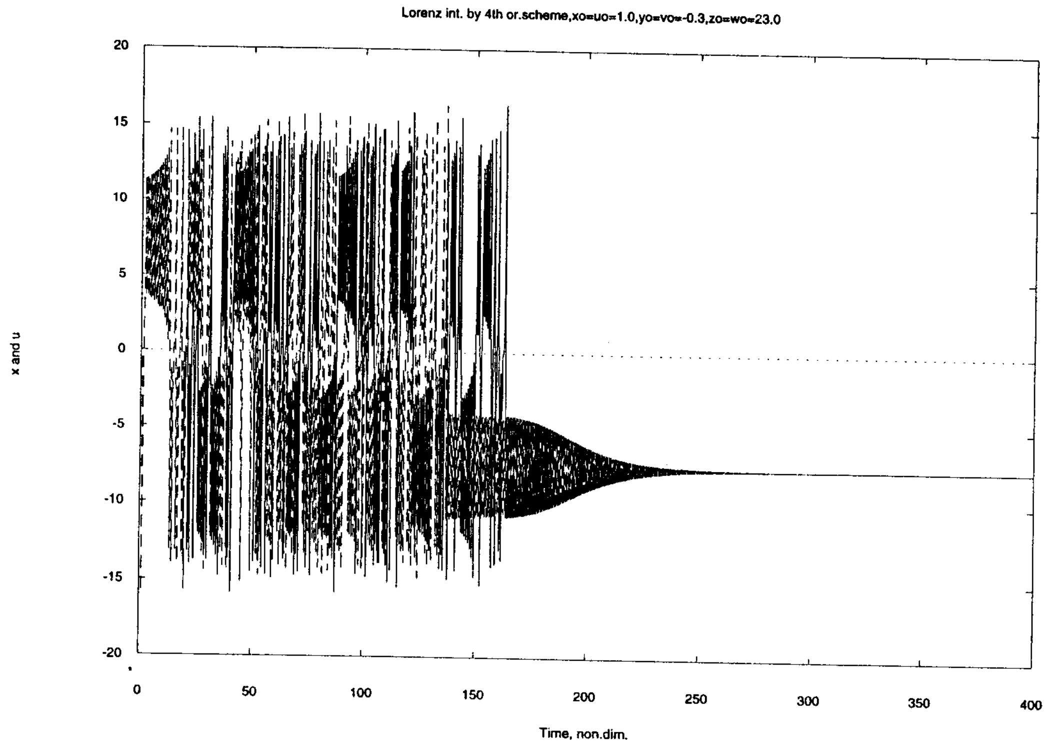

Fig. 4c. As Fig. 4a except that the common initial state is (1.0,-0.3, 23.0).

Additional integrations using a fourth order Runge-Kutta scheme with an adjustable time step and with a specified accuracy in each time step were carried out. The results of these integrations are summarized in Table 1.

Table 1

Accuracy	r = 23.0	r1 = 23.0001	r1 = 23.0000001
0.1	- - +	- - +	+ + +
0.01	+ + +	+ + +	- - +
0.001	+ + +	- - +	- - +
0.0001	+ + +	+ + +	+ + +
0.00001	+ + +	- - +	+ + +

The plus and minus signs in Table 1 are the signs of the dependent variables x, y and z in this order. It is far from easy to draw any conclusions from the table. The first column seems to indicate that for a given value of the Rayleigh number the increased accuracy leads to a choice of the steady state with positive values of x and y. However, the other two columns do not follow the same rule. For a specified accuracy of 10^{-4} we find the same steady state for all the three values of r, but the other three accuracies give different results. It is therefore likely that randomness determines the choice of one or the other of the two possibilities.

3. A second order stochastic version of the Lorenz model

Considering the results obtained in section 2 it appears that the model is particularly sensitive to changes in the initial state if the starting position is reasonable close to the unstable steady state (0,0,0), or the trajectory comes close to this point. In this situation it may be an advantage to use a stochastic model as designed originally by Epstein (1969). We shall adopt a model in which third and higher order statistics are neglected. This means that the equations will contain the variances σ (xx), σ(yy), σ(zz), σ(xy), σ(xz) and σ(yz) in addition to the mean values that also in this case will be denoted x, y and z. The total system will thus contain the nine equations given in (3.1), (3.2) and (3.3).

$$\frac{dx}{dt} = s(y - x)$$
$$\frac{dy}{dt} = -xz + rx - y - \sigma(xz) \tag{3.1}$$
$$\frac{dz}{dt} = xy - bz + \sigma(xy)$$

$$\frac{d\sigma(xx)}{dt} = 2s(\sigma(xy) - \sigma(xx))$$
$$\frac{d\sigma(yy)}{dt} = -2(z\sigma(xy) + x\sigma(yz) - r\sigma(xy) + \sigma(yy)) \tag{3.2}$$
$$\frac{d\sigma(zz)}{dt} = 2(y\sigma(xz) + x\sigma(yz) - b\sigma(zz))$$

$$\frac{d\sigma(xy)}{dt} = (r - z)\sigma(xx) + s\sigma(yy) - (1 + s)\sigma(xy) - x\sigma(xz)$$
$$\frac{d\sigma(xz)}{dt} = s\sigma(yz) - (s + b)\sigma(xz) + y\sigma(xx) + x\sigma(xy) \tag{3.3}$$
$$\frac{d\sigma(yz)}{dt} = (r - z)\sigma(xz) - (1 + b)\sigma(yz) + y\sigma(xy) + x(\sigma(yy) - \sigma(zz))$$

These equations reduce to the original equations for the Lorenz attractor, if all the variances are zero. The original steady states are thus steady states of the stochastic equations as well, but with all variances equal to zero. In addition, one may search for additional steady states present in the stochastic system. While the general case will be treated we shall begin with some special cases.

Special case I:

We start by considering the special case in which x = y = 0. By solving the nine steady state equations in this case it is seen that a steady state is $x = y = 0$, $z = r - 1$, $\sigma(xx) = \sigma(yy) = \sigma(xy) = b(r-1)$ and $\sigma(zz) = \sigma(xz) = \sigma(yz) = 0$. The stability of the new steady state may be found by solving the normal eigenvalue problem. The eigenvalues are most conveniently determined by numerical procedures. The stability was found as a function of the Rayleigh number, r, with the result that the steady state is stable for subcritical values of the Rayleigh number, i.e. for $r < r_c$, but otherwise unstable.

Special case II:

During the consideration of the general case (to be described later) it was discovered that one of the steady states have the values: $x = y = \frac{1}{2}[b(r-1)]^{\frac{1}{2}}$, and $z = \frac{1}{2}(r-1)$, i.e. half of the values for the deterministic case. With this information the equations can be solved for all the variances. The result is that $\sigma(xx) = \sigma(yy) = \sigma(xy) = b(r-1)$ and $\sigma(zz) = \sigma(xz) = \sigma(yz) = 0$. The investigation of the stability of this special steady state is that it is unstable for subcritical values of the Rayleigh number. Figure 5 shows the largest eigenvalue as a function of the Rayleigh number.

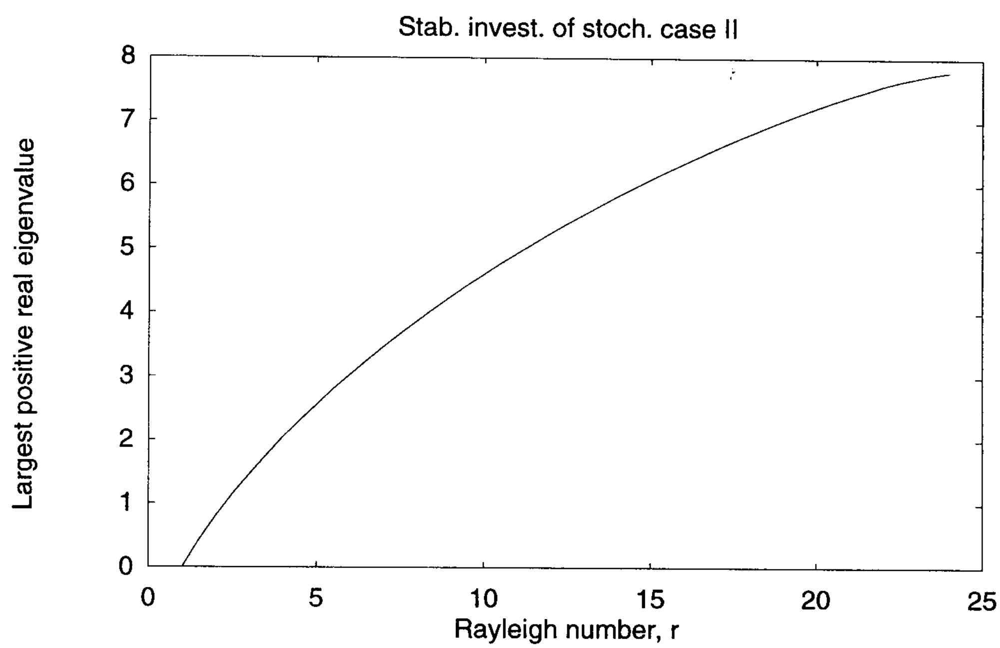

Fig. 5. The largest positive eigenvalue for case II as a function of r.

General case:

This case is the third kind of steady state present in the stochastic equations. For all possible steady states it is required that $x = y$ and $\sigma(xy) = \sigma(xx)$ as can be seen from the first equation in (3.1) and the first equation in (3.2). By a process of elimination, requiring a considerable amount of algebra it is possible to reduce the remaining seven steady state equations to two nonlinear coupled equations with the variables x and z. From one of these equations it is possible to express x^2 in terms of z. Substituting this expression in the second nonlinear equation a single equation, containing only the variable z, is obtained. It is solved by numerical procedures.

For certain values of the Rayleigh number three steady states are found. One of these is the steady state for the deterministic equations. It is unstable for supercritical and stable for subcritical values of the Rayleigh number. The second solution is the one treated as special case II. A third solution is obtained for $r \geq 6$. No analytical expression has been found for this solution. The numerical steady state values for the nine dependent variables have been used for various values of r to obtain all the eigenvalues by numerical procedures. The maximum eigenvalues are shown in Figure 6 as a function of r. It is seen that the steady state is unstable.

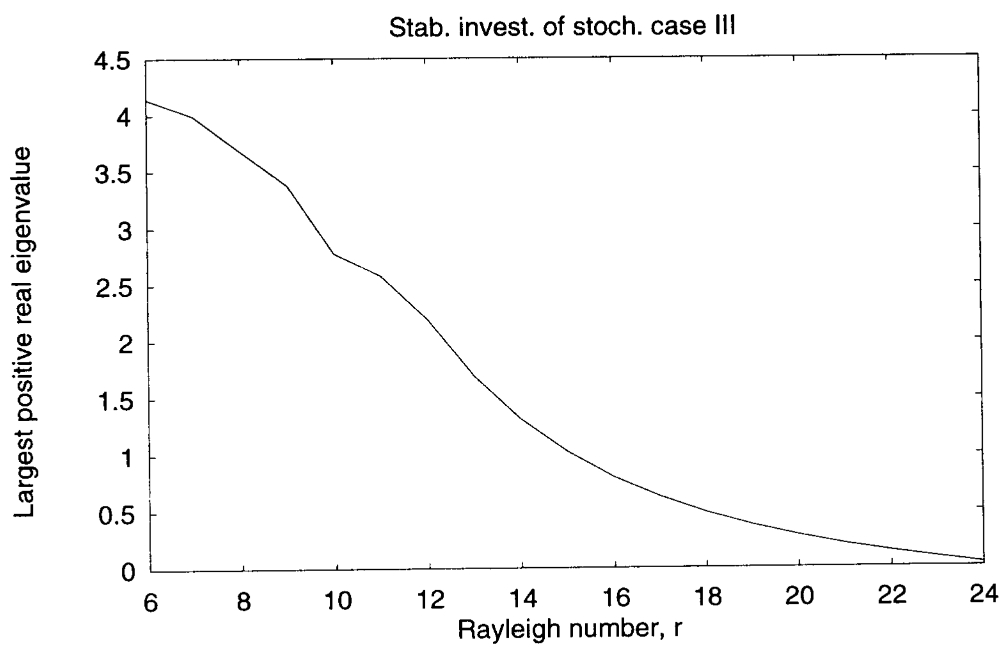

Fig. 6. The largest positive eigenvalue for the general case as a function of r.

In summary, we may conclude that the stochastic model has three additional steady state solutions of which only the very special solution having $x = y = 0$ is stable in the sub-critical domain. Since it is this domain that have our interest in the present investigation, we have introduced a new stable steady state. In addition, the unstable solutions will change the nature of the attractor at least in the immediate environment of them in the nine-dimensional space.

4. Comparison with a system without closure

Pitcher (1974) and Wiin-Nielsen (1982) have pointed out that on occasion it is possible to obtain solutions of a stochastic-dynamic problem without introducing a closure scheme. Such a solution rests on a Lagrangian technique and on the use of the conservation law for probabilities. The result can be expressed in the form:

$$E\left[f(x)\right] = \int f(x(x_o, t))\phi(x_o, t_o)dx_o \tag{4.1}$$

in which E is the expected value of the function f, while **x** is the vector of all the dependent variables. In the present case **x** = (x,y,z). The function φ may be selected in various ways. We shall as in the former cases use a Gaussian distribution. In the following the bar represents a mean value and the subscript o an initial value. Using (4.1) we shall calculate the asymptotic values (marked by subscript a) of the relevant quantities in the stochastic-dynamic model of the Lorenz attractor introduced in section 3. Using the same technique as developed for this problem by Wiin-Nielsen (1982) we find the results given in eq. (4.2) in which Φ is the error function as normally defined.

$$
\begin{aligned}
\bar{x}_a &= (b(r-1))^{1/2}\,\Phi\!\left(\frac{x_o}{(2\sigma_o(xx))^{1/2}}\right)\\
\bar{y}_a &= (b(r-1))^{1/2}\,\Phi\!\left(\frac{y_o}{(2\sigma_o(yy))^{1/2}}\right)\\
\bar{z}_o &= r-1\ .
\end{aligned}
\qquad (4.2)
$$

The reason for the appearance of the error function in the expressions for the mean values of x and y, but not in the mean value of z is that the asymptotic values of x and y may have both positive and negative values under the integration, while the asymptotic value of z always is r-1. Note, however, that the formulas given in (4.3) may be used for both negative and positive values of x_o and y_o because Φ(-x) = - Φ(x).

Among the variances we find that σ (xx), σ(yy) and σ (xy) only are different from zero. The formulas for these variances are given in eq. (4.3).

We shall next make a comparison between the results obtained from the stochastic-dynamic model in section 3 and the results which can be calculated from eq. (4.3). The stochastic-dynamic model with its neglect of third and higher order moments puts the user in an either-or situation in the sense that the asymptotic state will either be (0,0,r-1) or one of the states $(\pm(b(r-1))^{1/2}, \pm(b(r-1))^{1/2}, r-1)$. The estimates made in the present section will result in new steady state for each choice of the initial values of x, y and the three variancies. To make the comparison easier we shall restrict the initial uncertainties to a choice of σ (xx) = σ (yy).

$$
\begin{aligned}
\sigma(xx) &= b(r-1)\left(1-\Phi^2\!\left(\frac{x_o}{(2\sigma_o(xx))^{1/2}}\right)\right)\\
\sigma(yy) &= b(r-1)\left(1-\Phi^2\!\left(\frac{y_o}{(2\sigma_o(yy))^{1/2}}\right)\right)\\
\sigma(xy) &= b(r-1)\left(1-\Phi\!\left(\frac{x_o}{(2\sigma_o(xx))^{1/2}}\right)\Phi\!\left(\frac{y_o}{(2\sigma_o(yy))^{1/2}}\right)\right)\ .
\end{aligned}
\qquad (4.3)
$$

The results in eq. (4.3) indicate that if the initial values are selected in such a way that the argument in the error function is numerically large (say, larger than 2) then the asymptotic state will be close to one of the normal steady states of the Lorenz attractor, while a choice leading to a small value of the argument will give a solution close to the special steady state on the z-axis, i.e. (0,0,r-1), and such a solution will

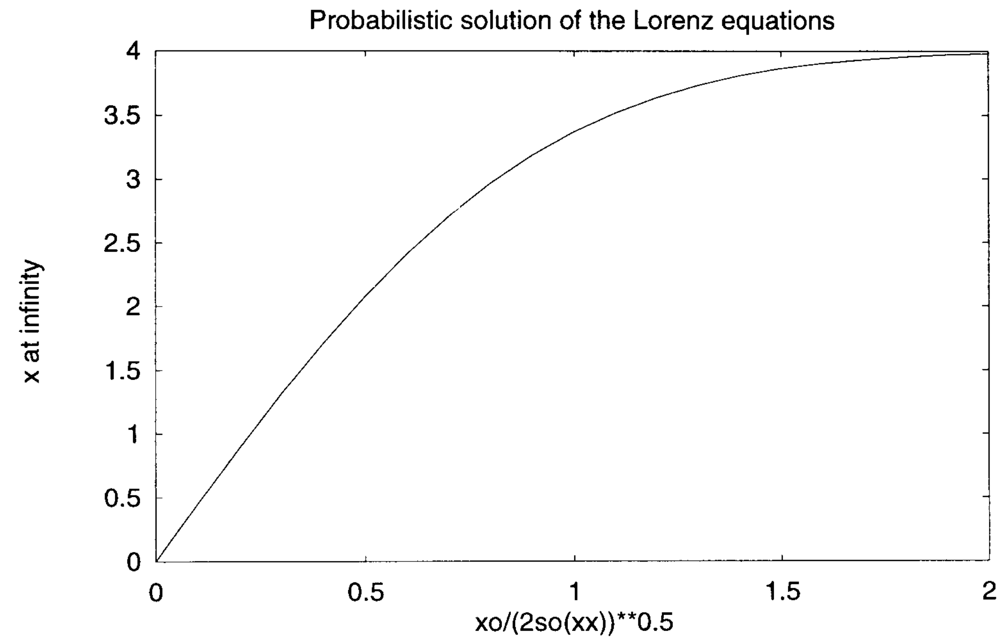

Fig. 7. The probabilistic solution for the asymptotic value of x (and y).

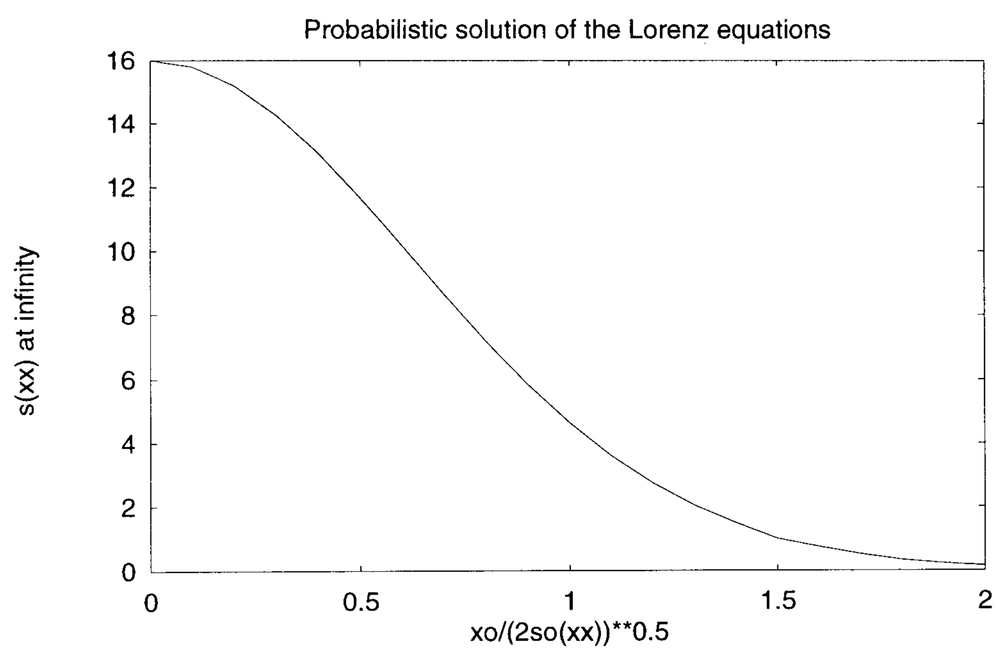

Fig. 8. The probabilistic solution for the asymptotic value of σ (xx) (and σ (yy), σ(xy)).

furthermore give values of the variances close to $\sigma (xx) = \sigma (yy) = \sigma (xy) = b(r-1)$. Intermediate values of the argument in the error function will give a solution between these two extremes. To illustrate these statements Figure 7 shows the asymptotic value of x as a function of the argument of the error function, i.e. $\xi = x_o/(2\sigma\,(xx))^{½}$. It is seen that the solution is close to the solution of the Lorenz equations when the argument approaches 2. The same curve illustrates the asymptotic value of y. Figure 8 shows the asymptotic value of σ (xx) as a function of the argument of the error function. The curve illustrates at the same time the behavior of σ (yy) and σ (xy) under the restrictive conditions that have been used here for the initial uncertainties.

It is naturally of interest to determine for which value of the abscissa (ξ) of the error function one finds a change in the asymptotic state. Such a determination can be obtained only by numerical experiments. A couple of examples follow. For the initial state of $x_o = y_o = z_o = 0.1$ it turns out that the change takes place for values of σ (xx) = σ (yy) = σ (zz) between 0.179 and 0.180, while $x_o = y_o = z_o = 1.0$ lead to a change, when the variances are between 1.72 and 1.73.

5. Concluding remarks

The behavior of the integrations of the Lorenz-equations for subcritical values of the Rayleigh number is an example of limited predictability. The model shows high sensitivity to small changes in the forcing and/or the initial state. The asymptotic state is one of the two stable steady states. The selection of one of these for a given initial state seems to be random. The behavior of the uncertainties in the initial state has been simulated by designing a stochastic-dynamic model including only second order statistics. In addition to the steady states in the deterministic Lorenz model the stochastic dynamic equations contain three additional steady states of which only one is stable. This steady state is: (0,0,r-1).

By numerical experimentation starting in a given point (x,y,z) and with variances σ (xx), σ(yy) and σ(zz) it is demonstrated by numerical experiments that sufficiently small variances will lead to an asymptotic state identical with one of the stable steady states of the deterministic Lorenz equations, while larger values of the variances result in an asymptotic state of (0,0,r-1).

REFERENCES

Epstein, E. S., 1969. Stochastic dynamic prediction, *Tellus*, **21**, 739-759.

Epstein, E. S. and E. J. Pitcher, 1972. Stochastic analysis of meteorological fields, *Jour. of Atmos. Sci*., **29**, 244-257.

Fleming, R. J., 1971. On stochastic-dynamic prediction, Part I: The energetics of uncertainty and the question of closure, *Mo. Wea. Rev*., **99**, 851-872.

Lorenz, E. N., 1963. Deterministic nonperiodic flow, *Jour. of Atmos*. Sci., **20**, 130-141.

Pitcher, E. J., 1974. Stochastic dynamic prediction using atmospheric data, Ph.D. dissertation, Univ. of Michigan, 151 pp.

Wiin-Nielsen, A., 1982. On simple stochastic-dynamic systems with forcing, *Bulgarian Geophys. Jour*, **8**, 3-16.