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Steady state and transient solutions of the nonlinear forced shallow water equations in one space dimension

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RESUMEN

Las ecuaciones unidimensionales de agua poco profunda admiten soluciones de estado permanente para un forzamiento dado que es independiente del tiempo. Cuando el forzamiento es suficientemente pequeño se obtienen tres soluciones periódicas, de las cuales dos tienen velocidades numéricas altas y la tercera una velocidad un poco más baja. Se dan ejemplos de soluciones para patrones sencillos de forzamiento.

Las ecuaciones unidimensionales no lineales se simplifican de la manera usual, despreciando el término advectivo en la ecuación de continuidad y reemplazando el geopotencial con una constante cuando no está diferenciado. Se muestra que estas ecuaciones tienen solamente una solución de estado permanente que es semejante a la solución de baja velocidad en el sistema más general. Las aproximaciones actúan así como un filtro de soluciones de alta velocidad.

Las soluciones de las ecuaciones simplificadas se obtienen por formulación de las ecuaciones en el espacio de número de onda. Los ejemplos indican que un máximo número de onda de 30 es suficiente para obtener soluciones con bastante exactitud.

ABSTRACT

The one-dimensional shallow water equations permit steady state solutions for a given forcing that is independent of time. When the forcing is sufficiently small one obtains three periodic solution of which two have numerically large velocities and the third a somewhat lower velocity. Examples of solutions for simple forcing patterns are given.

The nonlinear one-dimensional equations are simplified in the usual way by neglecting the advection term in the continuity equation and by replacing the geopotential by a constant when undifferentiated. It is shown that these equations have only one steady state solution which is similar to the low velocity solution in the more general system. The approximations act thus as a filter of large velocity solutions.

Solutions of the simplified equations are obtained by formulating the equations in wave number space. Examples indicate that a maximum wave number of 30 is sufficient to obtain solutions of sufficient accuracy.

1. Introduction

The shallow water equations have played an important role in understanding the interplay between gravity waves and quasi-geostrophic waves since the basic investigations by Charney (1948) and Rossby (1949). In this connection reference is also made to the historical review by Lewis (1998). These problems have been treated without explicit use of forcing and dissipation and have given important contributions to the understanding of geostrophic adjustment and to the formulation of quasi-geostrophic models. In the present paper we shall add forcing and later also dissipation to the problem. It will be shown that steady state solution of the nonlinear equations can be obtained in some simplified cases. The treatment of a similar problem using the primitive equations and a specified heating is much more complicated. The general idea is described below.

Imagine a long channel of constant depth filled with a homogeneous fluid or a gas to a certain level. It is assumed that conditions at one end are similar to those at the other end, or, in other words, that one may assume periodic boundary conditions. In some sections fluid is then added, while fluid is removed in other sections, and in such a way that the net-addition of fluid is zero at all times. The specified addition and subtraction of fluid will create motion and changes in the depth of the fluid. If the forcing is kept constant in time as we shall assume in each case, it is conceivable that the fluid after a while will come to a steady state. The first task of the present investigation is to find the possible steady states.

2. The problem and the steady state solutions

The standard textbook perturbation treatment of wave motion in a homogeneous fluid with a free surface shows that one-dimensional gravity waves move with speeds of $U \pm \Phi^{1/2}$, where U is the constant zonal velocity and Φ is the constant geopotential in the basic state. It is quite a different matter to consider the possible steady state of a homogeneous fluid with a free surface when the fluid is forced as described above.

Under the assumptions mentioned in the introduction the proper equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \Phi\frac{\partial u}{\partial x} + S(x) \tag{2.1}$$

In these equations u is the velocity along the channel, φ the geopotential and $S(x)$ is the specified forcing. Note that we so far have neglected dissipation. This neglect is permissible because we have both positive and negative forcing, but is done here for convenience in order to solve the steady state equations. It is an advantage for our purpose to write the equations in the form:

$$\frac{\partial u}{\partial t} + \frac{\partial (1/2u^2 + \phi)}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u)}{\partial x} = S(x) \tag{2.2}$$

Neglecting the local time derivatives in both equations it is seen from the first equation that the sum of the kinetic and the potential energy remain constant. We shall assume that we start to add fluid when the velocities are zero everywhere, and when the fluid has a constant geopotential Φ_o. It is then everywhere true that $1/2\ u^2 + \Phi = \Phi_o$. The second steady state equation may be integrated to give:

$$\phi u = \int_o^x S(x')dx' = I(x) \tag{2.3}$$

$I(x)$ is a function that may be calculated from the given forcing function $S(x)$. The integrated first equation is multiplied by u, and we obtain the cubic equation given in eq. (2.4).

$$u^3 - 2\Phi_o u + 2I(x) = 0 \tag{2.4}$$

It should be noted that in deriving the formula for $I(x)$ we have assumed that the product $(u_o\varphi_o)$ vanishes. For one of the solutions u_o is zero. For the other two solutions where u_o is different from zero we calculate the geopotential for $x = 0$ and find that the geopotential vanishes in these two wawe solutions. The assumption is therefore justified.

Eq. (2.4) is a cubic equation that should be solved locally for each value of x in the interval under consideration. It is well known that (2.4) may have three real solutions if the discriminant (D) is negative. Two of these solutions will coincide if $D = 0$. Finally, if $D > 0$ there will be one real and two complex solutions. The discriminant is in our case given in (2.5).

$$D = I(x)^2 - \frac{8}{27}\Phi_o^3 \tag{2.5}$$

It is seen that D will be negative if the absolute value of the forcing everywhere is kept at a sufficiently low level. In such a case we have three real solutions.

On the other hand, the forcing may also be locally large giving a positive value of D in some points and not in others. In this case there will be no real steady state solution for the entire interval since we want continuous solutions. Assuming that $S = S_o \sin(kx)$ we find $I(x) = (S_o/k)2\sin^2(1/2kx)$. The discriminant is negative if S_o satisfies the inequality

$$S_o < (\frac{2}{27})1/2\Phi_o^{3/2}\frac{2\pi}{L} \tag{2.6}$$

Setting $L = 1x10^6 m$ and $\Phi_o = 2x10^3 m^2 s^{-2}$, corresponding to a depth of about 200 m, we find that the upper limit for S_o is about 0.15 $m^2\ s^{-3}$ corresponding to an addition of fluid at the rate of 1.5 cm s^{-1}. The limit of S_o must be calculated for each specification of the forcing. Using the same parameters as above, but replacing the sine function with a cosine function it turns out that the upper limit is twice as large.

Figure 1 shows the distribution of the three velocities for a case in which the parameters have been set to the following values: $L = 1000$ km, $S_o = 0.25\ m^2\ s^{-3}$, $\Phi_o = 2.0x10^3\ m^2\ s^{-2}$, and $S(x) = S_o \cos(kx)$. This selection of parameters gives three steady states everywhere.

Figure 2a shows the forcing function calculated in such a way that it has a maximum at a quarter of the total wavelength with zero values at $x = 0$ and $x = L$. In addition it has been required that the mean value of S vanishes. We use $L = 5000$ km, $S_o = 0.02\ m^2\ s^{-3}$ and $\Phi_o = 2000\ m^2\ s^{-2}$. Figures 2b, 2c and 2d shows the three velocities.

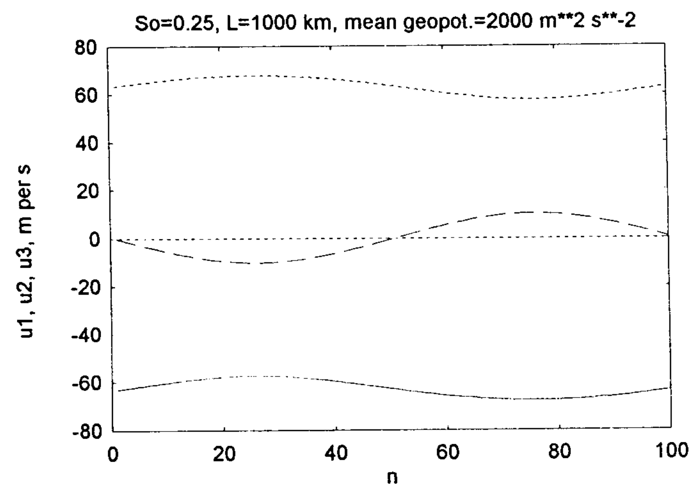

Fig. 1. The three wave solutions for $L = 1000$ km, $\Phi_o = 2x10^3$ m^2 s^{-2} and $S_o = 0.25$.

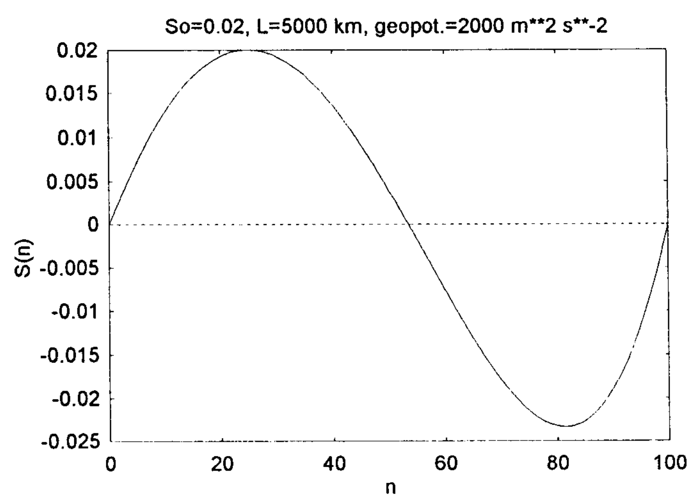

Fig. 2a. Forcing function with a maximum at $n = 25$, $L = 5000$ km and $\Phi_o = 2x10^3$ m^{2}s^{-2}.

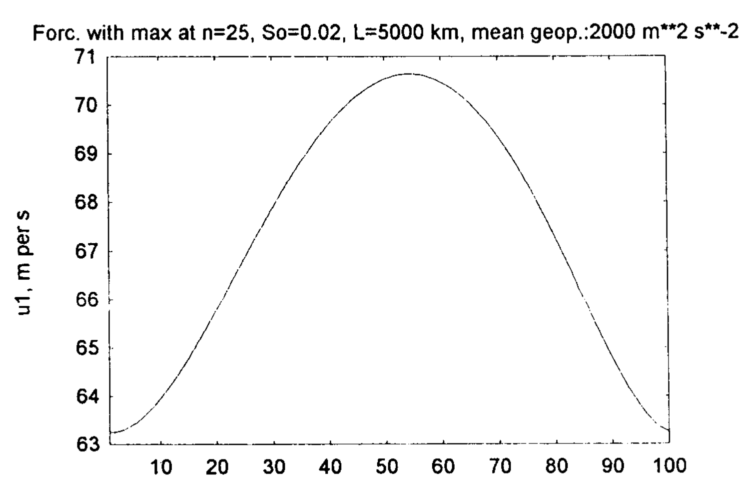

Fig. 2b. The velocity u_1 in m per s for the parameters given in 2a.

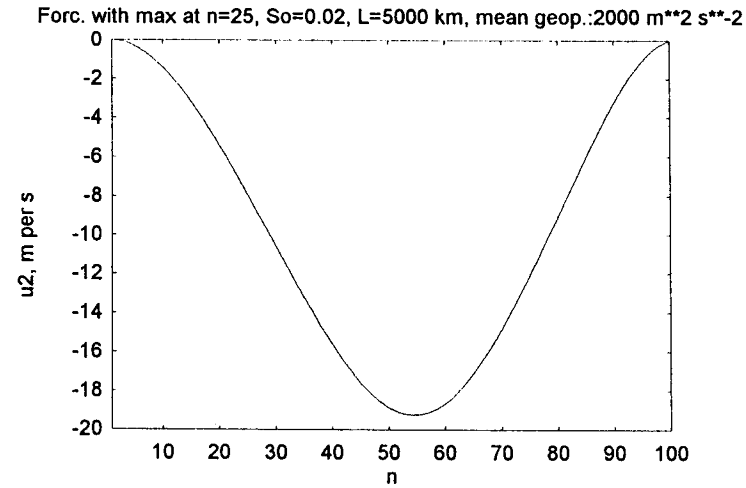

Fig. 2c. The velocity u_2 in m per s for the parameters given in 2a.

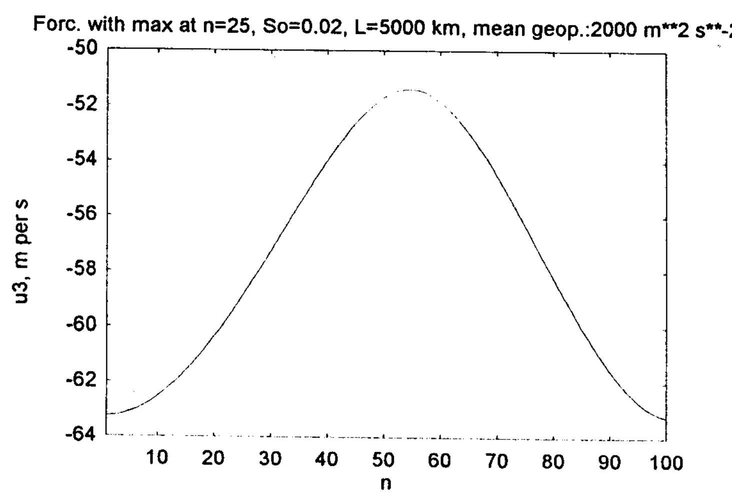

Fig. 2d. The velocity u_3 in m per s for the parameters given in 2a.

The next example has rather extreme values. The forcing has the same shape as in the previous case with $S_o = 0.01$ (see Figure 3a), but the total length has been selected as low as 10 m, while the mean value of the geopotential is 10 m^2 s^{-2} corresponding to a mean depth of approximately 1 m. Figures 3b, 3c and 3d show the much smaller velocities obtained in a case that could be used as a laboratory experiment.

It should be stressed that the solutions considered so far are general in the sense that no further assumptions have been introduced beyond the single space dimension (x). A common version of the equations is, however, to neglect the advection term in the continuity equation and to approximate the geopotential in the divergence term by a constant value. When these approximations are introduced the system (2.1) is changed to (2.7).

$$\frac{\partial u}{\partial t} + \frac{\partial[1/2u^2 + \phi]}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial t} + \Phi_o \frac{\partial u}{\partial x} = S(x) \tag{2.7}$$

The steady state solution of the system (2.7) are given in (2.8).

$$u = \frac{1}{\Phi_o} I(x)$$

$$\phi = \Phi_o - \frac{u^2}{2} \tag{2.8}$$

The approximations permit only one steady state. The remaining solution corresponds to the solution with the numerically smallest value. In addition, it is noted that this solution may be obtained from the cubic equation in (2.4) by assuming that the first term is small compared to the second, while a first approximation to the fast waves is obtained by neglecting the third term compared to the other two.

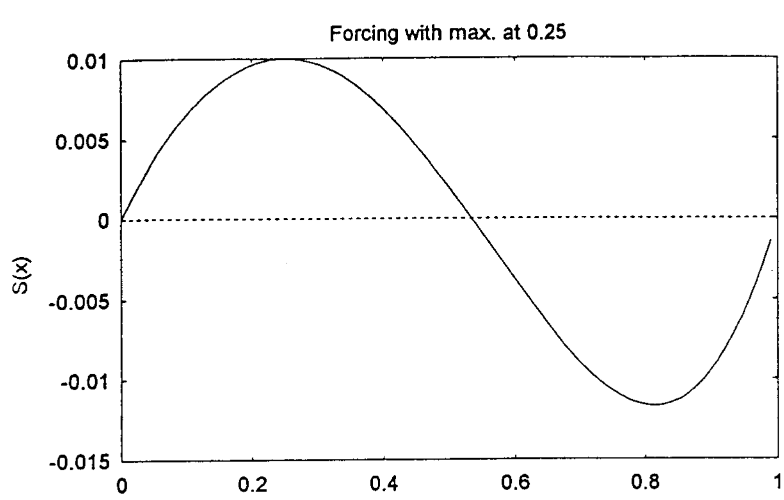

Fig. 3a. The same type of forcing as in Fig. 2a, but for $L = 10$ m, $\Phi = 10$ m^2 s$_{-2}$.

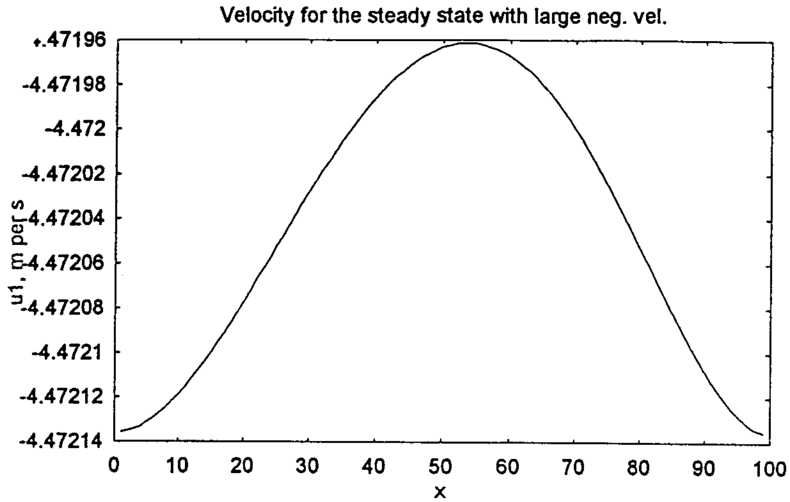

Fig. 3b. u_1 as a function of n (or x).

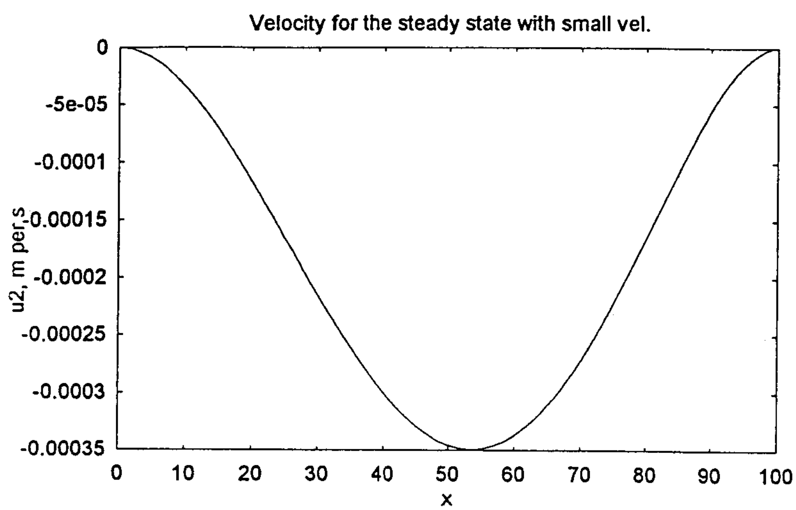

Fig. 3c. u_2 as a function of n.

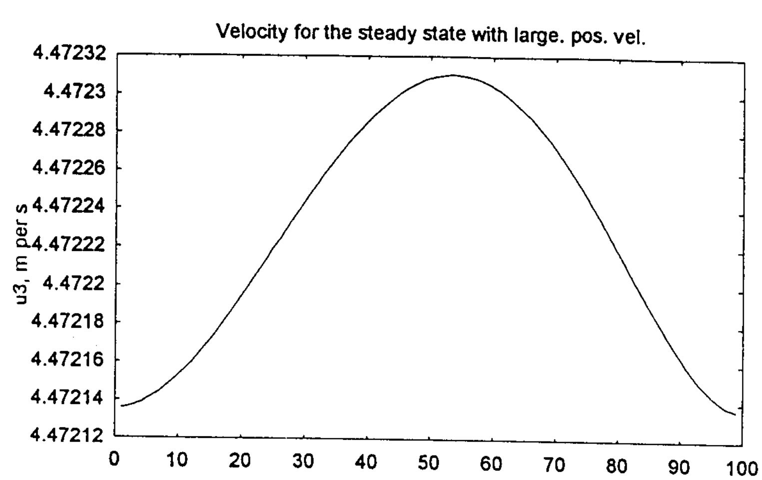

Fig. 3d. u_3 as a function of n.

3. The simplified system

The general system considered in section 2 is seldom used. It can be integrated using finite differences, but while it is energetically consistent (Wiin-Nielsen and Chen, 1993, see Chapter 2) with respect to the conversion from available potential to kinetic energy, it has an energy conversion which contains a triple integrand due to the fact that the homogeneous fluid has a surface varying with the horizontal coordinates. The energy conversion is in the one-dimensional case:

$$C(A,\ K) = \frac{\rho}{2gL}\int_{o}^{L} \phi^2 \frac{\partial u}{\partial x} dx \tag{3.1}$$

The kinetic energy itself is also an integral with a triple product in the integrand, i.e.

$$K = \frac{\rho}{1gL}\int_{o}^{L} u^2 \phi dx \tag{3.2}$$

while the available potential energy contains a quadratic integrand. Due to the form of the kinetic energy it has not been possible to derive meaningful formulas for the energy generation and the dissipation of kinetic energy.

Since the total depth of the fluid in most applications is very large compared to fluctuations of the free surface, it is often assumed that the energy quantities may be calculated with an upper limit equal to the mean depth. In order to make this approximation consistent one should use the equations given in (2.7). The amount of kinetic energy and the energy conversion $C(A,K)$ are obtained from (3.1) and (3.2) by introducing Φ_o. At the same time we include friction in the equation of motion through a horizontal diffusion, see (3.3), or a dissipation proportional to the velocity.

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{\partial \phi}{\partial x} + \nu \frac{\partial_u^2}{\partial_x^2}$$

$$\frac{\partial \phi}{\partial t} = -\Phi_o \frac{\partial u}{\partial x} + S(x) \tag{3.3}$$

An advantage of the reduced system is that it is well suited for a spectral formulation. Such a formulation was done by incorporating both the cosine and the sine functions in the series. It turns out, however, that a significant simplification can be obtained by including only the sine functions in the specification of the velocity and only the cosine functions in the series for the geopotential and the source function. Consequently, we specify the series as given in (3.4).

$$S = \sum_{n=1}^{nmax} S[n]\cos(nkx)$$

$$\phi = \sum_{n=1}^{nmax} \phi[n]\cos(nkx)$$

$$u = \sum_{n=1}^{nmax} u[n]\sin(nkx) \tag{3.4}$$

Introduction of the series in the equations result in the spectral equations given in (3.5) (Wiin-Nielsen, 1979).

$$\frac{du[n]}{dt} = \frac{kn}{2} \sum_{q=1}^{nmax-q} u[q]u[n+q]$$

$$-\frac{k}{2} \sum_{q=1}^{n-1} qu[q]u[n-q] + nk\phi[n] - \nu n^2 k^2 u[n]$$

$$\frac{d\phi[n]}{dt} = -\phi_o nku[n] + S[n] \tag{3.5}$$

After these formalities we shall consider the steady state solution to the system (3.3). As expected only a single steady state is found. It is given in (3.6), in which the function $I(x)$ is defined earlier. The solution is identical to the single root in the cubic equation (2.4) if the cubic term is neglected.

$$U_s = \frac{I(x)}{\Phi_o}; \Phi_s = \nu \frac{S(x)}{\Phi_o} - \frac{I(x)^2}{w\Phi_o^2} \tag{3.6}$$

The energy equations for the simplified system are given in symbolic equations in (3.7).

$$\frac{dA}{dt} = G(A) - C(A,\ K)$$

$$\frac{dK}{dt} = C(A,\ K) - D(K) \tag{3.7}$$

In our simplified case the generation of available potential energy, the conversion from this form of energy to kinetic energy and the dissipation of kinetic energy are given in (3.8).

$$G(A) = \frac{\rho}{g}\frac{1}{L} \int_o^L \phi S dx$$

$$C(A,\ K) = \frac{\rho\Phi_o}{g}\frac{1}{L} \int_o^L \phi \frac{\partial u}{\partial x} dx$$

$$D(K) = \frac{\nu\Phi_o\rho}{g}\frac{1}{L} \int_o^L (\frac{\partial u}{\partial x})^2 dx \tag{3.8}$$

In the case of the steady state solution given above it follows that $G(A) = C(A, K) = D(K)$. This statement was checked by calculating the three quantities from the steady state solution separately for a simple specification of $S(x) = S_o \cos(kx)$. The dissipation integral is convenient to work with. It is $D(K) = (\rho\nu S_o^2)/(2g\Phi_o\ )$ in which ρ is the constant density of the fluid. Assuming that we apply the model to the atmosphere and use $\rho = 1$ kg/m^3, $\Phi_o = 10^5$ m^2/s^2, $\nu = 10^5$ m^2 s^{-2} we find for $S_o = 0.1$ m^2/s^3 that $D(K) = 5x10^{-4}$ W/m^2 or a very small intensity of the circulation, while the intensity of the steady state circulation become 0.05 W/m^2 if the value of S_o becomes 10 times larger.

In addition to the steady states we shall also look at a numerical integration of the spectral equations given in (3.5). The starting point is the forcing $S(x)$ shown in Figure 4. The rather simple configurations of $S(x)$ in the present and the following example are copied from Hodgman (1963, see pages 378). It has $S_o = 0.02$ m^2 s^{-3}, $\Phi_o = 1.0x10^5$ m^2 s^{-2}, $\nu = 1.0x10^{-5}$ m^2 s^{-1} and $L = 2.8x10^7$ m, where all numbers are given in standard units. It should perhaps be mentioned that the value of S_o corresponds to a maximum value of 2 mm s^{-1} for the additon of fluid. The Fourier components of $S(x)$ was calculated and used as the forcing. Starting from an initial state of rest and with the constant value of the geopotential the spectral equations were integrated for 1000 days to obtain a good approximation to the asymptotic case. This integration gives thus a measure of the spin-up time and the information that the steady state is stable. Figure 5a shows the function $u = u(x)$ at the end of the integration and Figure 5b gives the geopotential at the same time. The spectral integration was carried out with $n_{max} = 30$.

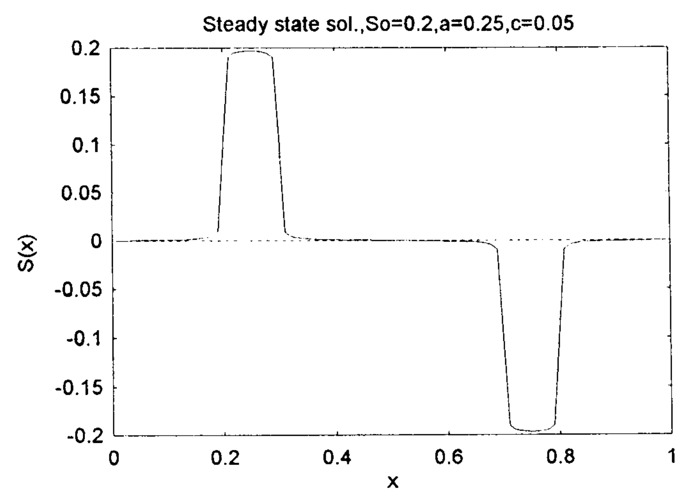

Fig. 4. $S(x)$ for $S_o = 0.2$, $a = 0.25$ and $c = 0.05$.

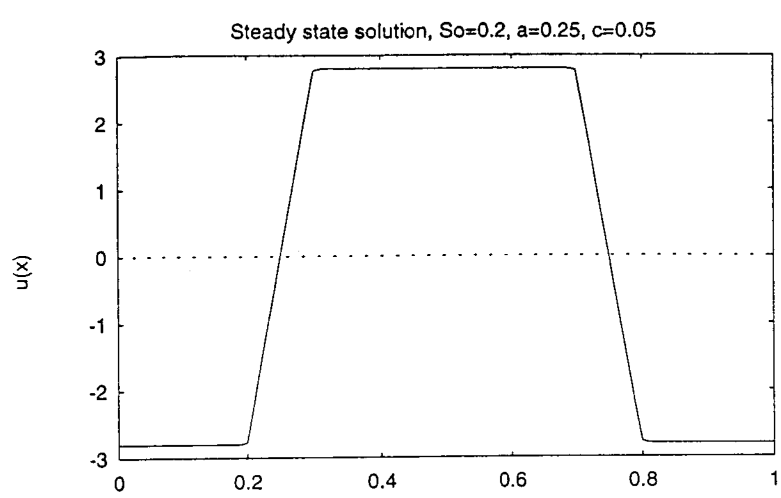

Fig. 5a. $u(x)$ for the parameters given in Fig. 4.

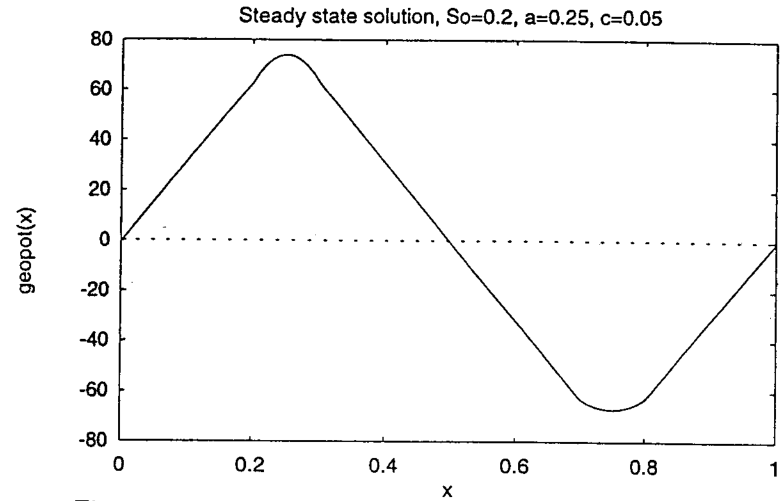

Fig. 5b. $\varphi(x)$ for the parameters given in Fig. 4.

A second example of a specification of $S(x)$ is shown in Figure 6. The corresponding steady state values of $u(x)$ are given in Figure 7a and the geopotential in Figure 7b. The maximum wave number is also in this case equal to 30.

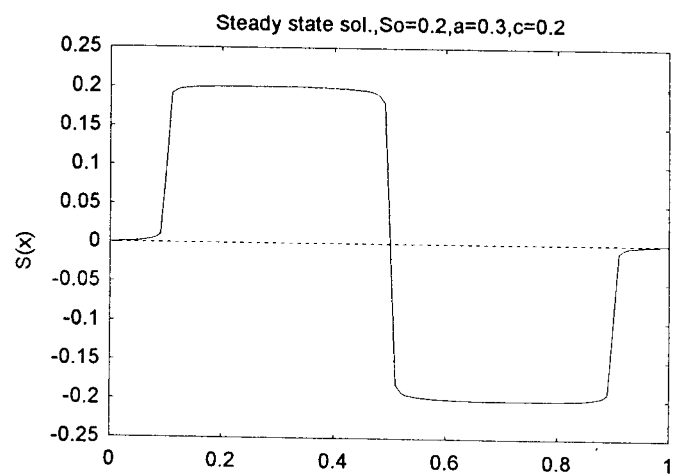

Fig. 6. $S(x)$ for $S_o = 0.2$, $a = 0.3$ and $c = 0.2$.

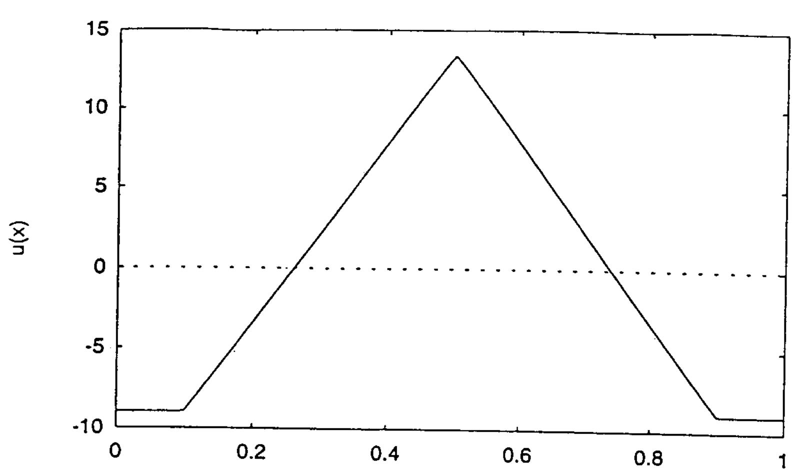

Fig. 7a. $u(x)$ for the parameters given in Fig. 6.

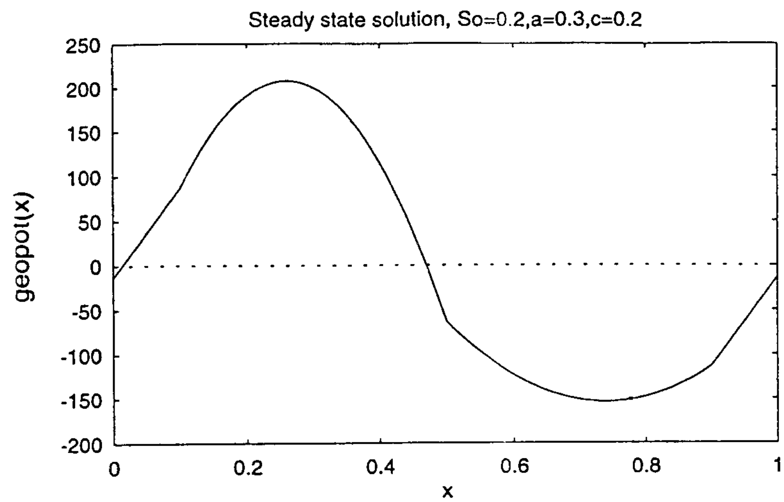

Fig. 7b. $\varphi(x)$ for the parameters given in Fig. 6.

An especially simple example is one for which the forcing is S_o in the left half of the region and thus $-S_o$ in the right half. Defining $\mu = x/L$ we find the following steady state:

$$u_L = A\mu; u_R = A(1-\mu); A\frac{S_o L}{\Phi_o}$$

$$\phi_L \Phi_o - 1/2A^2\mu^2; \phi_R = \Phi_o - 1/2A^2(1-\mu)^2. \tag{3.9}$$

The exact steady state may be compared with the steady state obtained by expressing both the forcing and the variables in wave number space. As before we use a maximum wave number of 30. Figure 8 compares the exact expression for the forcing with the forcing as expressed by the contributions from the components in the spectral domain. Although some small scale noise is

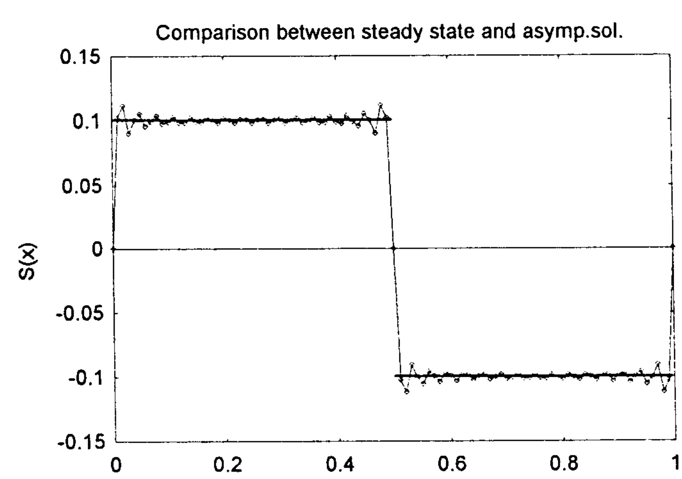

Fig. 8. Comparison between $S(x)$ given as $S = 0.1$ in the left half and $S = -0.1$ in the right half of the region and as the sum of the Fourier coefficients.

observed the main pattern are well approximated. Figure 9 makes a similar comparison between the u-velocities. An almost perfect agreement is obtained for the velocities. We may therefore assume that the other cases in which the forcing is given in wave number space represent good accuracy as well. We shall finally look at the results of a couple of numerical integrations. In the integrations the steady state was determined first. Thereafter the initial state was perturbed by 1% on all wave numbers. The first integration was conducted without any frictional force for 50 days. A single result is shown in Figure 10 which indicates that a small amplitude oscillation is found in $G(A)$. The period is about 7 days. Dissipation in the linear form was introduced in the second integration. Figure 11 shows that $G(A)$ in this case approaches a value of 1.33 which coincide with the value of $G(A)$ in the steady state. We may thus conclude that the steady state is stable.

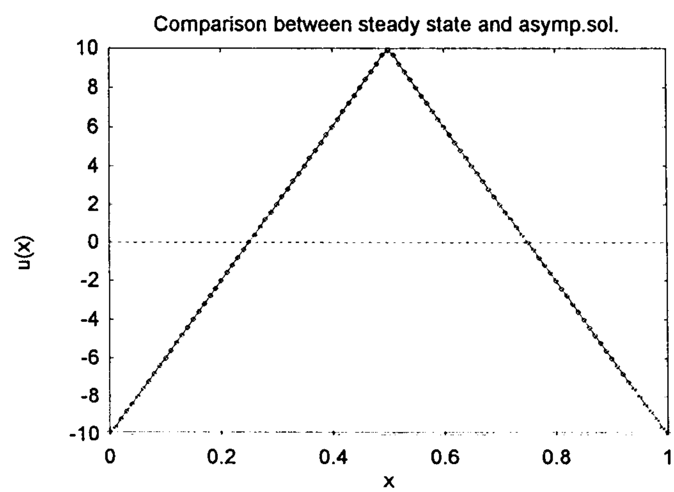

Fig. 9. Comparison of $u(x)$ as obtained analytically and as a sum of the Fourier series.

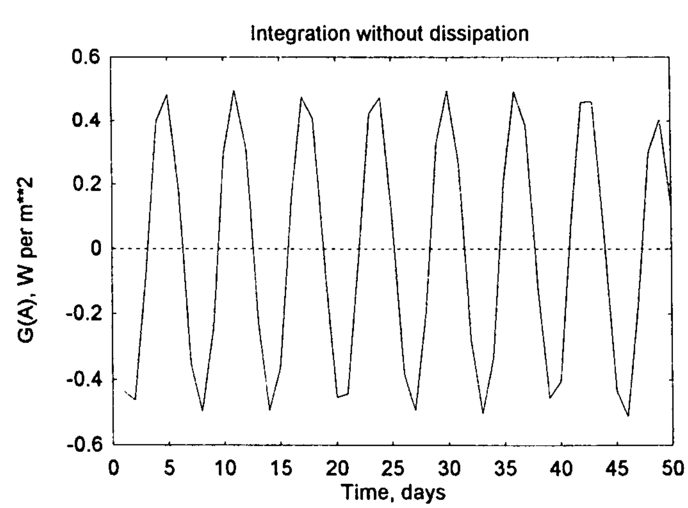

Fig. 10. The generation of available potential energy, $G(A)$, as a function of time for a case without dissipation.

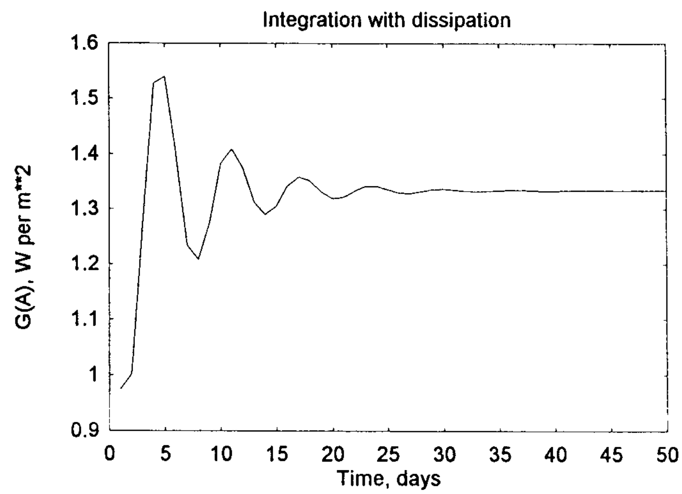

Fig. 11. The generation of available potential energy, $G(A)$, as a function of time for a case with linear dissipation ($\epsilon = 4x10^{-6}\ s^{-1}$).

4. Summary and concluding remarks

The main purpose of the present paper is to obtain stationary solutions of the nonlinear shallow water equations in one space dimension when fluid is added and subtracted in such a way that the net addition vanishes. In spite of the nonlinearity it is in this special case possible to obtain exact solutions for a given forcing that may vary in the space dimension, but not in time.

For the general equations in one space dimension one may obtain three solutions provided that the forcing is everywhere kept sufficiently low. The upper limit may be determined from the properties of the relevant cubic equation since this equation should have three real roots for all points in the interval under consideration. The velocities obtained in this case may be characterized by two solutions with numerically large velocities, but of opposite signs and a third solution with numerically smaller velocities.

The shallow water equations are simplified in the usual way by the neglect of the advection term in the continuity equation and by using a mean value for the geopotential when appearing in undifferentiated form. The resulting equations obtained in this way have for a given forcing one and only one solution. This solution compares well with the general solution resulting in the small velocities. We observe therefore that also in the forced case the approximations act as a filter eliminating phenomena with large speeds.

In the approximate equations it is often an advantage to express the forcing, the velocity and the geopotential in wave number space. Since it is necessary to select a maximum wave number for each determination of the steady state it is of importance to compare an exact solution with a solution in wave number space. Such a comparison has been made in the very special case in which fluid is added to the left half at a constant rate and removed at the same rate from the right half of the region under consideration. It appears that a maximum wave number of about 30 gives results of good accuracy.

The present system can in certain respects be considered as equivalent to a forcing of the atmosphere by a specified heating. The forcing as applied has the same dimension as a heating, i.e. $m^2\ s^{-3}$ or W per kg.

It will be desirable to expand the present investigation to include the effect of the Coriolis terms. This can be done using the quasi-geostrophic theory on the shallow water equations.

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