The zonal atmospheric structure: A heuristic theory, Part II

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RESUMEN

Se reconsidera el estado zonal de la atmósferasando coeficientes de interacción para simular la acción recíproca entre las ondas atmosféricas y el estado zonalmente promediado. Con el fin de corregir el viento del este demasiado grande en un modelo previo, los coeficientes de interacción se especifican en función de la latitud. El procedimiento revisado se aplica a un modelo acuoso poco profundo en el que la agregación de fluidos es análoga al calentamiento de la atmósfera. Con un forzamiento simétrico el geopotencial y la velocidad horizontal zonalmente promediada deberían ser también campos simétricos, mientras que la función de corriente y la vorticidad deberían ser antisimétricas. Los cálculos previos a los vientos zonalmente promediados fueron demasiado fuertes en la región cercana al ecuador aunque la dirección se determinó correctamente. Los cálculos presentados dan un resultado mejorado respecto a los vientos tropicales del este.

En las integraciones se ha supuesto que el transporte meridional de vorticidad puede ser descrito por coeficientes de interacción. Esta suposición hace posible determinar de una manera indirecta el transporte meridional del momentum.

ABSTRACT

The zonal state of the atmosphere is reconsidered using interaction coefficients to simulate the interplay between the atmospheric waves and the zonally averaged state. In order to correct too large easterly wind in an earlier model the interaction coefficients are specified as functions of latitude. The revised procedure is applied to a shallow water model in which addition of fluid is an analogue to the heating of the atmosphere. With a symmetric forcing the geopotential and the zonally averaged horizontal velocity should also be symmetric fields, while the streamfunction and the vorticity should be anti-symmetric. In the earlier calculations the zonally averaged winds were of too strong in the region close to the equator although the direction was determined in a correct way. The present calculations give an improved result with respect to the tropical easterlies.

In the integrations it has been assumed that the meridional transport of vorticity may be described by interaction coefficients. This assumption makes it possible to determine in an indirect way the meridional momentum transport.

1. Introduction

In an earlier paper (Wiin-Nielsen, 1994), hereafter referred to as Part I, a simple theory for the zonally-averaged state of the atmosphere was presented. While the theory gave qualitatively correct results and was able to account for both the increase of the zonal winds with height and the latitudinal distribution of easterlies and westerlies it produced in general too large easterly winds in the low latitudes. The present paper will show results from an improved formulation of the integrations, but this time we shall for simplicity use a model based on a incompressible and homogeneous fluid with a free surface.

The reader is referred to Part I for a general discussion of the historical aspects of the various theories that has been developed in the past and to a description of the parameterization of the interactions between the atmospheric waves and the zonally averaged state of the atmosphere. Part I contains also references to papers by authors who have given important contributions to the understanding of the shape and intensity of the zonally averaged state of the atmosphere. These references will not be repeated in the present paper.

Fully developed global models of the general circulation are able to account for the zonal structure of the atmosphere, and these models should of course be used if the desire is to account for all the details. It has been realized for a long time that the meridional transport of sensible and latent heat as well as the meridional transport of momentum by the atmospheric waves are essential for the zonal structure of the atmosphere. This study, as well as the investigation in Part I, attempts to parameterize these transports in such a way that they become expressible in zonally averaged quantities. The main principle is that the transports of quasi-conservative quantities may be expressed in terms of interaction coefficients.

Quasi-geostrophic or quasi-nondivergent models contain some problems with respect the general structure of the zonally-averaged state. The model in the original formulation was quasi-geostrophic as formulated by Charney (1948, 1949). Its application in short-range numerical weather prediction, using a formulation without forcing and dissipation, gave some difficulties due to the meridional variation of the Coriolis parameter. These difficulties were removed by the use of the so-called balance equation (Charney, 1955) giving the nonlinear relation between the geopotential and the streamfunction in a nondivergent flow. The model became then a quasi-nondivergent model.

When heating and dissipation are introduced in the model some difficulties arise if the model is used in the global, spherical domain. As an example one may consider a model with a heating that is symmetric around the equator. In that case the same symmetry will exist for the geopotential, the temperature, the zonally-averaged windfield and the vertical velocity, while the streamfunction and the vorticity will be asymmetric. However, the normal formulation of the potential vorticity equations uses the streamfunction (and the vorticities) which are asymmetric and the symmetric heating in undifferentiated form. If we use the equations in this form we will thus obtain symmetric fields of the streamfunction and thus asymmetry for those variables that should be symmetric.

One may circumvent these difficulties in several ways. One of them is to formulate the vorticity equations replacing the streamfunction by the geopotential in which case one has to assume that the Coriolis parameter is constant in the formula for the vorticity. The streamfunction is then obtained from a full or a simplified form of the balance equation. This procedure was used in Part I. Another possibility is to maintain the variable Coriolis parameter and multiply each term by it in which case the heating term will be multiplied by sine of the latitude and thus be converted from a symmetric to an anti-symmetric term. This procedure has been tried, but the results have so far been unsatisfactory.

The models used in Part I were based on the two level, quasi-nondivergent model with specified heating. In the investigation to be described in this note we shall simplify the model even further by applying a shallow water model in which the zonal forcing is simulated by adding and subtracting fluid from the model in such a way that the net-addition vanishes. The averaged depth of the fluid is thus kept constant. Such a model (Wiin-Nielsen, 1998) has recently been used to investigate atmospheric variations in the zonal direction. In the present investigation the model will be used to simulate the meridional variations.

2. The model

The model to be used in the following will be of the quasi-geostrophic kind as formulated originally by Charney (1948). We recall that a major result of his scale analysis was that for the large-scale motion one may assume that the advective winds are non-divergent, but in all other terms the divergence inherent in the model has to be considered.

The vorticity and continuity equations for the flow of an incompressible and homogeneous fluid are given in (2.1) and (2.2).

$$\frac{\partial \varsigma}{\partial t} + \vec{v}_* \cdot \nabla(\varsigma + f) = -f_0 \nabla \cdot \vec{v} - \epsilon \varsigma \tag{2.1}$$

$$\frac{\partial \phi}{\partial t} + \Phi_o \cdot \nabla \vec{v} = gS \tag{2.2}$$

In these equations we have used standard notations. The wind vector which according to the foundation of quasi-geostrophic model may be considered as non-divergent has a subscript * in eq. (2.1). The advection term does not appear in the continuity equation since the advection of the geopotential by a geostrophic wind is zero.

S is a forcing function measuring the amount of fluid added per unit time with the restriction that the space average of S vanishes. It is the analogue of the heating in the thermodynamic equation, and it is noted that gS has the same dimension as the heating per unit mass and unit time, i.e. J kg^{-1} s^{-1} = m^2 s^{-3}. We may eliminate the divergence between these two equations with the result given in (2.3).

$$\frac{\partial \varsigma}{\partial t} + \nabla \cdot ((\varsigma + f)\vec{v}) = \frac{f_0}{\Phi_o}(\frac{\partial \phi}{\partial t} - gS) - \epsilon \varsigma \tag{2.3}$$

The next step is to take the zonal average of (2.3) using the spherical coordinates. The radius of the Earth are denoted by a. We obtain the equation given in (2.4).

$$\frac{\partial}{\partial t}[\varsigma_z - \frac{f_0}{\Phi_o}\phi_z] = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}[\cos\varphi(\varsigma_e v_e)_z] - \frac{f_0}{\Phi_o}gS - \epsilon \varsigma_z \tag{2.4}$$

The first term on the right hand side of (2.4) contains the meridional transport of eddy vorticity. In accordance with the discussion in Part I we make the assumption that this transport may be approximated by an interaction coefficient as given in (2.5).

$$(\varsigma_e v_e)_z = -K\frac{\partial \varsigma_z}{a\partial \varphi} \tag{2.5}$$

We introduce the meridional variable $\mu = \sin(\varphi)$ and since we shall use examples in which S is

symmetric around the equator we approximate the vorticity by setting it equal to the Laplacian of the geopotential divided by a constant value of the Coriolis parameter. The equation to be integrated is thus given in (2.6).

$$\frac{\partial[L(\phi) - q^2\phi]}{\partial t} = \frac{\partial}{\partial \mu}[\frac{K}{a^2}(1 - \mu^2)\frac{\partial L(\phi)}{\partial \mu}]gq^2S - \epsilon L(\phi) \tag{2.6}$$

In (2.6) L means the Laplacian of a zonally averaged field. Note also that the subscript z has been dropped. In addition we have introduce the notation q^2 with a dimension of m^{-2} as given in (2.7).

$$q^2 = \frac{f_o^2}{\Phi_o} \tag{2.7}$$

The interaction coefficient K was assumed to be a constant in Part I, and it was found that this assumption resulted in too strong easterlies in the low latitudes. It appears therefore that this assumption leads to a too large export of momentum from the low latitudes to the middle latitudes. Some test integrations were made with an assumption that $K = 0$ from the equator to a latitude φ_* while being a constant from this latitude to the North Pole. While some improvement was noticed in the low latitudes, it was found that too large values were now observed at the very large latitudes. As a result of these test integrations it appears that an improved specification could be to assume that K should have its largest values in the middle latitudes where the wave generation is at a maximum, while small values of K should be at low and high latitudes. A relatively simple specification of such a variation in K is obtained by assuming that K has the following form:

$$K = K_m\mu^2(1 - \mu^2) \tag{2.8}$$

When (2.7) is non-dimensionalized by using a, the radius of the Earth, as the length scale and $f_o^{-1} = \Omega^{-1}$ as the time scale we find the equation given in (2.9).

$$\frac{\partial[L(\phi) - a^2q^2\phi]}{\partial t} = \hat{K}\frac{\partial}{\partial \mu}[\mu^2(1 - \mu^2)^2\frac{\partial L(\phi)}{\partial \mu}] - \hat{S} - \hat{\epsilon}L(\phi) \tag{2.9}$$

In Part I it was found convenient to transform the final linear equation into spectral form, and it was therefore easy to determine the steady state. (2.9) can of course also be brought into a spectral form, but due to the meridional variation of the interaction coefficient it becomes quite cumbersome to deduce the spectral equation. The reason is that terms such as $\mu^2 P_n$ and $\mu^4 P_n$ have to be expressed in terms of the Legendre polynomials. It is therefore more convenient to integrate (2.9) by using finite differences in space and time. In the integrations reported in the next section centered differences in time and space (with an uncentered time step at t=0) have been used. These integrations are carried out for a time necessary to reach a steady state. Experience shows that integrations have to be carried out for approximately 50 days to approach the steady state with excellent accuracy, if one starts from a state of rest.

After each time step we get a new value of the potential vorticity, say, $V(\mu)$, in each gridpoint. It is thus necessary to solve the Helmholtz equation

$$L(\phi) - a^2q^2\phi = V(\mu) \tag{2.10}$$

The solution could be obtained by the usual iterative scheme, but in the present case it was found more convenient to write $V(\mu)$ as a sum of Legendre polynomials, calculate the coefficients $V(n)$, obtain the Legendre coefficients for the Laplacian of the geopotential from (2.10) and obtain the distribution of the Laplacian or the geopotential with latitude by summing the series.

3. Results

For each integration it is necessary to specify the forcing S. Many possibilities are available, but we shall restrict S to have the general form given in (3.1) assuring symmetry around the equator.

$$S(\mu) = \sum_{n=1}^{n} s(n) \cos(n\pi\mu) \qquad (3.1)$$

In the first example we use $N = 1$ and $s(1) = 0.03$ m per s. The interval from equator to the North Pole is represented by 100 intervals making $d\mu = 0.01$ corresponding to a grid size of 100 km. The counter of the grid points is denoted by q running from 0 at the equator to 100 at the North Pole. The forcing is displayed in Figure 1 showing positive forcing in the southern half and negative forcing in the northern half of the hemisphere. The steady state geopotential is shown in Figure 2.
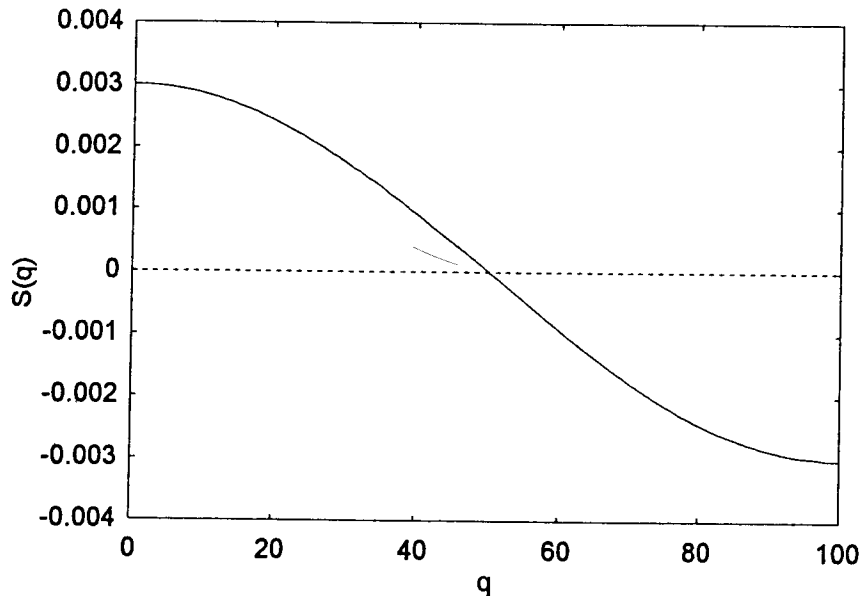

Fig. 1. The forcing in m per s as a function of the meridional counter, where $q = 0$ corresponds to the equator and $q = 100$ to the North Pole.

The zonal winds should be obtained from the streamfunction. For this purpose we use the full balance equation given in (3.2). Note that the most cumbersome terms vanish because we consider the zonally averaged flow.
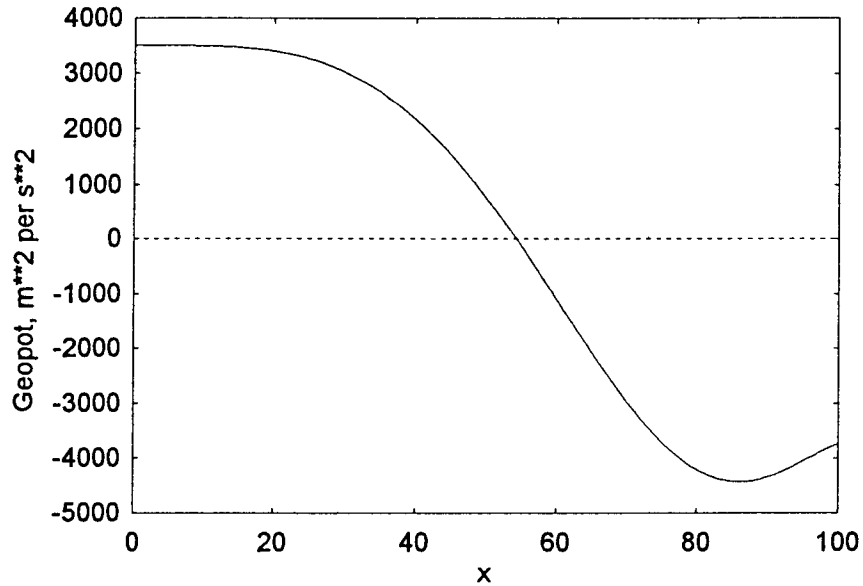
Fig. 2. The steady state geopotential as a function of q.

$$\nabla \cdot (2\Omega\mu\nabla\Psi) = \nabla^2\psi \qquad (3.2)$$

Eq. (3.2) is non-dimensionalized and solved by developing the variables in series of Legendre polynomials. Equating the coefficients for the same order of the Legendre polynomials we obtain a recursion relationship which may be written as given in (3.3).

$$\frac{n}{2n+3}\varsigma(n+1) + \frac{n+1}{2n-1}\varsigma(n-1) = \nabla^2(\phi(n)) \qquad (3.3)$$

The only disadvantage of (3.3) is that it does not permit the calculation of the first component of the vorticity because we would have to use $n = 0$ in that case. It is, however, seen that the coefficient of the first component in this special case is equal to zero. This problem was discussed at length in Part I. In the present case we have set the first component equal to zero, whereafter the odd components have been computed. The components of the vorticity may easily be converted to the components of the streamfunction ($\varsigma(n) = -(n+1)\psi(n)$), and it is then possible to calculate the zonal winds. For the present case the zonal winds are shown in Figure 3. It is seen that we obtain reasonable magnitudes of the equatorial easterlies, the mid-latitude westerlies and the polar easterlies.

The energy conversions may also be obtained. They are calculated using the steady state solution. In the present case we obtain $G(A_z) = C(A_z, K_z) = 6.6$ Wm^{-2}, $C(K_E, K_z) = -1.8$ Wm^{-2} and $D(K_z) = 4.8$ Wm^{-2}. As expected we find that the energy conversion is from KZ to KE due to the quasi-barotropic nature of the model. The conversions are of the same order of magnitude as found from observational studies (Wiin-Nielsen and Chen, 1993).

As a second example we use a case in which the forcing is given by $s(1) = 1.5S_o$, $s(2) = -0.1S_o$ and $s(3) = -0.4S_o$ with $S_o = 0.03$ m per s. This forcing is shown in Figure 4. The maximum forcing is now in the low latitudes and not at the equator as in the first case. Similarly, the largest negative forcing is at high latitudes, but not at the North Pole. The forcing arrangement gives a larger gradient in the forcing in middle latitudes. The geopotential for this case is given

in Figure 5 and shows also a larger gradient in the middle latitudes. The zonal winds, displayed in Figure 6, have a larger magnitude as should be expected, but the distribution is similar to the one shown in Figure 3.
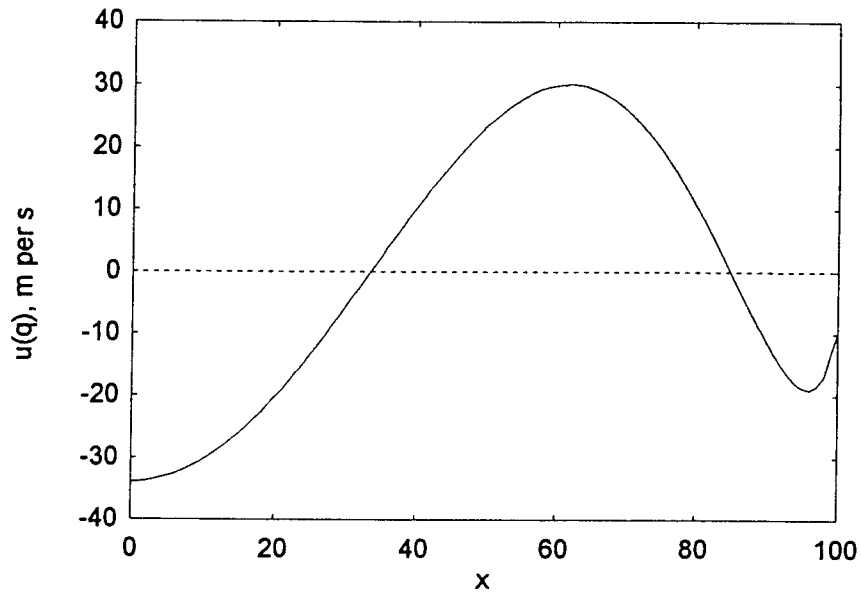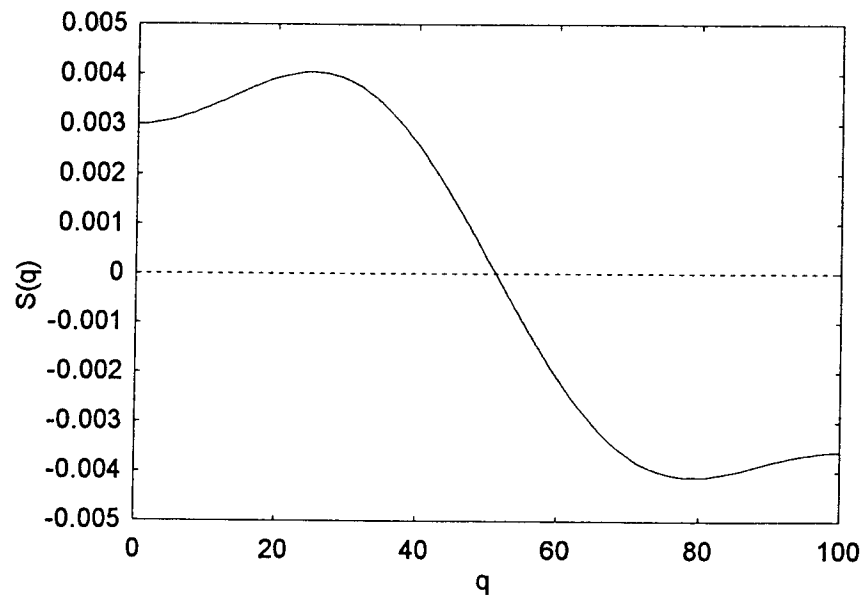
Fig. 3. The zonal velocity as a function of q.

Fig. 4. The zonal forcing in m per s as a function of q in the second example.
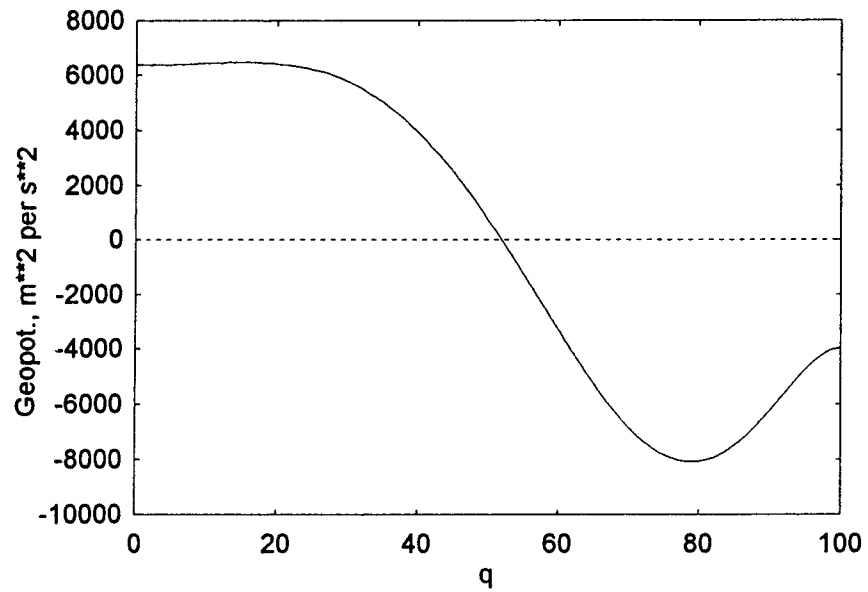
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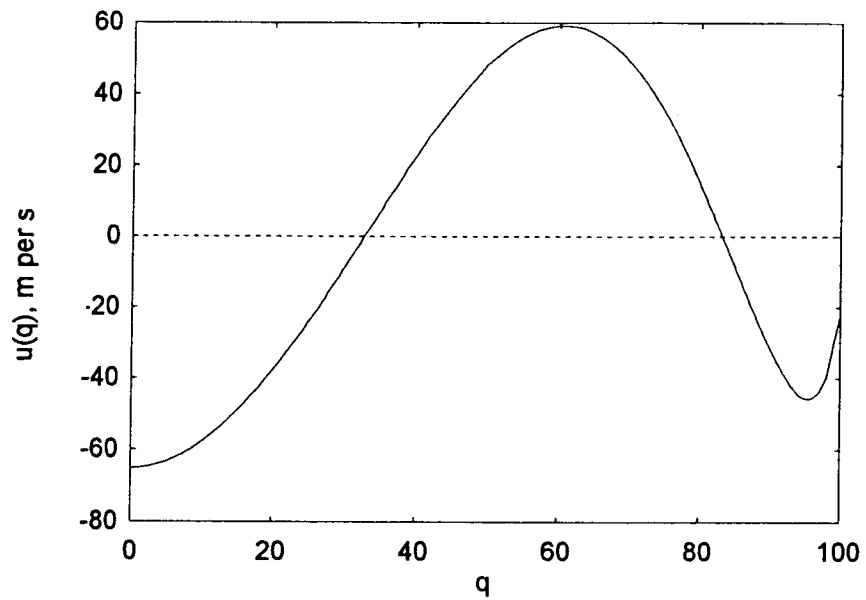Fig. 5. The geopotential as a function of q in the second example.

Fig. 6. The zonal velocity as a function of q in the second example.

In addition to the basic quantities described above it is possible to determine the momentum transport implied by the parameterization used in the model. In quasi-geostrophic model it is of course assumed that the advective velocity is nondivergent. It can then be shown that the transport of vorticity is related to the momentum transport. To determine the relationship in spherical coordinates it is an advantage to start from the vorticity transport, use the spherical formula for this quantity and the fact that the wind components are nondivergent. We get then:

$$(\varsigma v)_z = [-\frac{1}{a \cos \varphi} \frac{\partial u \cos(\varphi)}{\partial \varphi} v]_z$$

$$= -[\frac{1}{a \cos^2(\varphi)} \frac{\partial u \cos(\varphi)}{\partial \varphi}(v \cos \varphi)]_z \qquad (3.4)$$

The expression in (3.4) is integrated by parts. The last expression can then be written as follows:

$$-\frac{1}{a \cos^2(\varphi)} \frac{\partial (uv)_z \cos^2(\varphi)}{\partial \varphi} - \frac{1}{a \cos^2(\varphi)}[u \cos(\varphi) \frac{\partial v \cos(\varphi)}{\partial \varphi}]_z \qquad (3.5)$$

The last term in (3.5) will, however, give no contribution, because the flow is assumed to be nondivergent. This condition leads to the equation:

$$\frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos(\varphi))}{\partial \varphi} = 0 \qquad (3.6)$$

and it is then obvious that the last term averages to zero.

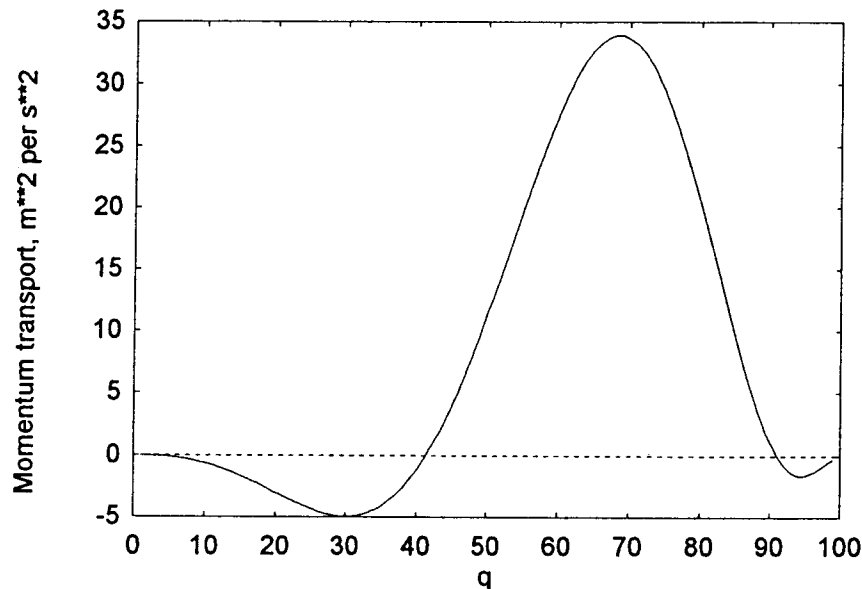The momentum transport has been calculated for the second example and is shown in Figure 7.

Fig. 7. The momentum transport in m^2 s^{-2} as a function of q in the second example.

4. Conclusions

The shallow water equation applied in a zonally averaged form and with the introduction of an interaction coefficient varying with latitude provides a simple explanation of the distribution of the zonal wind with latitude. The resulting winds are in good agreements with observations and have a correct order of magnitude. The simplicity of the model prevents a description of the variation of the zonal winds with height, and the model has no transport of sensible heat. These aspects of the general circulation will require a baroclinic model with at least two levels.

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