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Tests of a numerical algorithm for the linear instability study of flows on a sphere

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RESUMEN

Ha sido desarrollado un algoritmo numérico para estudiar la inestabilidad lineal de un flujo estacionario no divergente en una esfera en rotación. La precisión del algoritmo se prueba con soluciones zonales de la ecuación no lineal de vorticidad barotrópica (polinomios de Legendre, ondas zonales Rossby-Harwitz y modones monopolares).

ABSTRACT

A numerical algorithm for the normal mode instability study of a steady nondivergent flow on a rotating sphere is developed. The algorithm accuracy is tested with zonal solutions of the nonlinear barotropic vorticity equation (Legendre polynomials, zonal Rossby-Harwitz waves and monopole modons).

1. Introduction

The instability caused by the existence of a sufficiently large horizontal shear of the basic flow is known as barotropic instability (Pedlosky, 1979). The barotropic instability of atmospheric flows has been studied for half a century (Kuo, 1949; Lorenz, 1972; Simmons *et al.*, 1983; Haarsma and Opsteegh, 1988; etc.). Lorenz (1972) noted that this type of instability accounted for the existence of a limit in the atmospheric predictability. On the other hand, Simmons *et al.* (1983) showed that barotropic instability can be responsible for a low frequency variability of the barotropic atmosphere. Usually a flow is said to be stable if any its perturbation remains bounded with time in some norm. The two most commonly used types of the flow stability are: the linear (exponential or algebraic) stability when the flow is subjected to infinitesimal perturbations (Demidovich, 1967) and the nonlinear stability when the flow is subjected to small but finite perturbations (Liapunov, 1950, 1966; Arnold, 1965).

Many papers have been devoted to the linear instability of the zonal flows since the famous work by Rayleigh (1880) (see, for example, Pedlosky, 1979; Drazin and Reid, 1981; Green, 1995). However, there are a lot of important questions that remain to be answered. The application of numerical methods permits further insight into this problem. However, the basic challenge is the accuracy of the numerical stability results (Skiba, 1998). Therefore, testing of numerical algorithms used in the stability study is of paramount importance. In the present work we describe a numerical algorithm developed for the normal mode (exponential) instability of arbitrary steady flow in an ideal nondivergent fluid on a rotating sphere. In the first stage of the test we analyze only zonal solutions (Legendre polynomial flows, zonal Rossby-Haurwitz waves and monopole modons) of the equation

$$\frac{\Delta\psi}{\partial t} + J(\psi, \Delta\psi + 2\mu) = 0 \tag{1}$$

It is the nonlinear barotropic vorticity equation (BVE) written here in a non-dimensional form in the spherical coordinates (λ, μ) where λ is the longitude, $\mu = \sin\phi$, ϕ is the latitude, ψ is the streamfunction, Δ is the Laplacian, and J is the Jacobian (Silberman, 1954; Adem, 1956; Platzman, 1962). The stability of a steady solution to problem (1) is of considerable hydrodynamic and geophysical interest. It should be noted that only the exponential (normal mode) stability can be analyzed numerically, whilst computer calculations are unusable in the algebraic stability study. Indeed, every nondefective matrix can be diagonalized by a similarity transformation. Therefore, the algebraic growth of perturbations occurs just in the case when the stability matrix is defective, that is, when its Jordan canonical form has at least one Jordan block of dimension $k \times k$ where $k > 1$ (Stewart and Sun, 1990; Demidovich, 1967). Since the Jordan canonical form is too sensitive, and can be destroyed even by an infinitesimal perturbation of the elements of the stability matrix (Wilkinson, 1965), the algebraic instability cannot be analyzed numerically. The numerical stability study algorithm has been tested here using a few theoretical normal mode stability results obtained for exact solutions to the BVE (1). In particular, we use the Rayleigh-Kuo necessary condition for the instability (Rayleigh, 1880; Kuo, 1949) which has been almost the only useful instability condition known for zonal solutions to the BVE for half a century. Notice that the utility of the Rayleigh-Kuo condition is limited, since each Legendre polynomial (LP) flow $aP_n^0(\mu)$ (being a basic function for the zonal flows) satisfies this condition if $n \geq 3$ and amplitude a is sufficiently large (Baines, 1976; Skiba, 2000). We do not use the Tung (1981) instability condition, since its application in practice is not easier than that of the famous Liapunov (1966) theorems; instead, we apply the necessary normal mode instability conditions recently obtained for the LP flows, steady Rossby-Haurwitz waves and modons in Skiba (2000) and Skiba and Strelkov (2000a,b). These conditions are related to the spectral structure of unstable normal modes and state that Fjörtoft's (1953) average spectral number of the amplitude of an unstable mode must be equal to a special number. For a LP flow $aP_n^0(\mu)$, the new instability condition complements the Rayleigh-Kuo condition in the sense that while the latter deals with the basic flow structure; the former characterizes the structure of a growing perturbation. Also, we have checked the stability of the LP flows with $n = 1$ and $n = 2$ (Baines, 1976) and taken into account that zonal wavenumber m of unstable modes of each LP flow of degree n must satisfy the condition $0 < |m| < n$ (Skiba, 1989; Skiba and Adem, 1998). All these assertions are given in the next section (see Cases 1-5).

2. Normal mode stability method

We now describe the normal mode method for the study of the exponential instability of a two-dimensional steady flow on a rotating sphere (for more details about the normal mode method on a sphere, see Hoskins (1973), Baines (1976), Simmons *et al.* (1983), Dymnikov and Skiba (1987), Skiba (1989), Verkley (1990), Skiba and Adem (1998)). Let $\widetilde{\psi}(\lambda,\mu)$ be the streamfunction of a solution to equation (1) on the unit sphere S. The evolution of an infinitesimal perturbation $\psi(\lambda,\ \mu,\ t)$ to $\widetilde{\psi}(\lambda,\ \mu)$ is described by

$$\frac{\partial\xi}{\partial t} = \mathcal{L}\xi$$

$$\xi(\lambda,\ \mu, t_0) = \xi_0(\lambda,\mu) \tag{2}$$

where $\xi = \Delta\psi$ is the perturbation vorticity,

$$\mathcal{L}\xi = -J(\widetilde{\psi},\xi) - J(\Delta^{-1}\xi,\widetilde{\Omega}) \tag{3}$$

is the linear operator defined on sufficiently smooth complex-valued functions, and $\widetilde{\Omega} = \Delta\widetilde{\psi} + 2\mu$ is the absolute vorticity of the basic flow. The inner product and the norm of functions on S are defined as

$$\langle f,g\rangle = \int_S fg^* ds \equiv \int_{-1}^{1}\int_0^{2\pi} fg^* d\lambda d\mu$$

and

$$\|f\| = \langle f,f\rangle^{1/2}$$

respectively, where g^* is the complex conjugate of g. We will use the spectral method (Silberman, 1954; Machenauer, 1979) when both the solution $\widetilde{\psi}$ and the perturbation ψ are expanded on S in an infinite series of the orthonormal spherical harmonics $Y_n^m(\lambda,\mu) = P_n^m(\mu)e^{im\lambda}$ where $P_n^m(\mu)$ is the associated Legendre function of degree n and zonal wavenumber m . We denote by $\mathcal{P}^N$ a subspace of spherical polynomials of the degree N generated by $N(N+2)$ harmonics $Y_n^m(\lambda,\mu)$ with $-n \le m \le n$ and $n = 1, 2, ..., N$. In order to construct the matrix representing the operator $\mathcal{L}$ of problem (2)-(3) in the subspace $\mathcal{P}^N$, we assume that $\widetilde{\psi}(\lambda,\mu)$ belongs to $\mathcal{P}^M$, and $\psi(\lambda,\mu,t)$ and $\xi(\lambda,\mu,t) = \Delta\psi(\lambda,\mu,t)$ belong to $\mathcal{P}^N$ $(M < N)$, that is,

$$\tilde{\psi} = \sum_\beta^M \tilde{\psi}_\beta Y_\beta\ , \quad \psi = \Delta^{-1}\xi = -\sum_\alpha^N \chi_\alpha^{-1}\xi_\alpha Y_\alpha\ , \quad \xi = \sum_\alpha^{N_\alpha} \xi_\alpha Y_\alpha \tag{4}$$

For the sake of simplicity, hereinafter we use the following generally accepted notations: $\alpha \equiv (m_\alpha, n_\alpha)$, $Y_\alpha \equiv P_{n_\alpha}^{m_\alpha}(\mu)e^{im_\alpha\lambda}$, $\chi_\alpha \equiv n_\alpha(n_\alpha+1)$, $\xi_\alpha \equiv \xi_{n_\alpha}^{m_\alpha}$, and

$$\sum_\alpha^N \equiv \sum_{n_\alpha=1}^N \sum_{m_\alpha=-n_\alpha}^{n_\alpha}$$

(Platzman, 1962; Merilees, 1968; Hoskins, 1973; Machenauer, 1979; Skiba, 1989; Pérez-García and Skiba, 1999). Substituting (4) in (2), (3) and taking the inner product of the equation obtained with a spherical harmonic Y_α leads to

$$\frac{d}{dt}\xi_\alpha = \langle \mathcal{L}\xi, Y_\alpha \rangle = \int_S (\mathcal{L}\xi) Y_\alpha^* ds = \sum_\gamma^N L_{\alpha\gamma}\xi_\gamma \tag{5}$$

where

$$L_{\alpha\gamma} = \langle \mathcal{L}Y_\gamma, Y_\alpha \rangle \tag{6}$$

is the (α, γ)-element of the matrix L representing operator $\mathcal{L}$ in the subspace $\mathcal{P}^N$. Substituting (3) in (6) gives

$$L_{\alpha\gamma} = -\langle J(\tilde{\psi} + \chi_\gamma^{-1}\tilde{\Omega}, Y_\gamma), Y_\alpha \rangle \tag{7}$$

Thus, in subspace $\mathcal{P}^N$, problem (2) is reduced to

$$\frac{d}{dt}\vec{\xi} = L\vec{\xi} \tag{8}$$

The matrix element (7) can be written as

$$L_{\alpha\gamma} = \sum_\beta^M B_{\beta\alpha\gamma}\tilde{\xi}_\beta + D_{\alpha\gamma} \tag{9}$$

(Skiba, 1989) where $D_{\alpha\gamma} = i2m_\gamma\chi_\gamma^{-1}\delta_{\alpha\gamma}$ is the pure imaginary diagonal element of L, since $\delta_{\alpha\gamma} \equiv \delta_{n_\alpha n_\gamma}\delta_{m_\alpha m_\gamma}$ is the product of two Kronecker deltas, and

$$B_{\beta\alpha\gamma} = i(\chi_\beta^{-1} - \chi_\gamma^{-1}) \int_{-1}^{1} \frac{1}{(1-\mu^2)} P_\alpha(\mu) \left[m_\beta P_\beta(\mu) H_\gamma(\mu) - m_\gamma P_\gamma(\mu) H_\beta(\mu) \right] d\mu$$

is the triad nonlinear interaction coefficient, calculated with Gaussian quadrature formula (Machenauer, 1979). Looking for infinitesimal perturbation (4) in the form of a normal mode

$$\psi(\lambda, \mu, t) = \Psi(\lambda, \mu)e^{\omega t}\ , \quad \xi(\lambda, \mu, t) = \Delta\Psi(\lambda, \mu)e^{\omega t} \tag{10}$$

leads to the spectral problem

$$\mathcal{L}V = \omega V$$

for the operator $\mathcal{L}$ where $\omega = \omega_r + i\omega_i$ is the eigenvalue, and $V(\lambda, \mu) = \Delta\Psi(\lambda, \mu)$ is the corresponding eigenfunction. The streamfunction of a normal mode (10) can be written as

$$\psi(\lambda, \mu, t) = |\Psi(\lambda, \mu)| e^{\omega_r t} e^{i(\theta + \omega_i t)} \tag{11}$$

where

$$\Psi(\lambda, \mu) = \Psi_r(\lambda, \mu) + i\Psi_i(\lambda, \mu) = |\Psi(\lambda, \mu)| e^{i\theta}$$

is the amplitude, and $\theta \equiv \theta(\lambda,\mu) = \arg\Psi(\lambda,\mu) = \arctan\{\Psi_i(\lambda,\mu)/\Psi_r(\lambda,\mu)\}$ is the initial phase of the mode. Thus, a mode (10) is unstable if $\omega_r > 0$, decaying if $\omega_r < 0$, neutral if $\omega_r = 0$, and stationary if $\omega_i = 0$. The e-folding time τ_e and period T of the mode are determined (in days) as $\tau_e = \frac{1}{2\pi|\omega_r|}$ and $T = \frac{1}{|\omega_i|}$, respectively.

The differential spectral problem $\mathcal{L}V = \omega V$ is reduced to

$$L\overrightarrow{v} = \omega\overrightarrow{v} \tag{12}$$

in the subspace $\mathcal{P}^N$ where $\overrightarrow{v} = \{V_\gamma\} = \{-\chi_\gamma\Psi_\gamma\}$ is the eigenvector whose components are Fourier coefficients V_γ of the eigenfunction V with $n_\gamma \leq N$.

3. Instability conditions

We will test the numerical normal mode stability algorithm with three types of zonal flows being exact BVE solutions: LP flows, zonal Rossby-Haurwitz waves and monopole modons.

a) Legendre-polynomial (LP) flow:

$$\tilde{\psi}(\mu) = aP_n^0(\mu) \tag{13}$$

This zonal flow is described by the single Legendre polynomial of degree $n \geq 1$ and amplitude a.

b) *Zonal Rossby-Haurwitz (RH) wave:*

$$\tilde{\psi}(\mu) = -\frac{2}{(\chi_n - 2)}\mu + aP_n^0(\mu) \tag{14}$$

Here

$$\chi_n = n(n+1), \tag{15}$$

$n > 1$, and $a \neq 0$ is arbitrary real amplitude. It is a particular (zonal) case of the stationary RH wave

$$\tilde{\psi}(\lambda,\mu) = -\frac{2}{(\chi_n - 2)}\mu + \sum_{m=-n}^{n} \tilde{\psi}_n^m Y_n^m(\lambda,\mu) \tag{16}$$

being an exact solution to the BVE (1) for arbitrary coefficients $\tilde{\psi}_n^m$.

c) Monopole modon with uniform absolute vorticity in the inner region :

$$\tilde{\psi}(\mu) = \begin{cases} B_o S_\sigma^0(\theta) - C_o\mu + D_o \text{ in } S_o \\ B_i T^0(\theta) - C_i\mu + D_i \text{ in } S_i \end{cases} \tag{17}$$

(Verkley, 1990). This zonal flow is defined in different ways in two regions of the sphere S: the inner modon region $S_i = \{(\lambda, \mu) \in S : \mu > \mu_a\}$ and outer modon region $S_o = \{(\lambda, \mu) \in S : \mu < \mu_a\}$, $0 < \mu_a < 1$. Here $C_o = 2/(\chi_\sigma - 2)$; $C_i = 2/(\chi_\alpha - 2)$; B_o, B_i, D_o and D_i are constants; $\mu = \mu_a$ is the boundary between the regions S_i and S_o; $S_\sigma^0(\theta) = P_\sigma^0(-\cos\theta)$; θ is the colatitude; $T^0(\theta) = \ln\cos\left(\frac{\theta}{2}\right)$.

Note that for a zonal flow, the structure of mode (11) is

$$\psi(\lambda, \mu, t) = \Psi(\mu) e^{im\lambda} e^{\omega t} \tag{18}$$

In trials of the numerical normal mode stability study algorithm we have used the following theoretical results:

Case 1. (Kuo, 1949). Let $\tilde{\psi}(\mu)$ be a zonal flow on S. Then a normal mode (18) may be unstable only if the derivative $\frac{d}{d\mu}\tilde{\Omega}$ of the absolute vorticity $\tilde{\Omega} = \Delta\tilde{\psi} + 2\mu$ of flow $\tilde{\psi}(\mu)$ changes its sign at least in one point of the interval (-1,1).

For LP flow (13), $\frac{d}{d\mu}\tilde{\Omega} = 2 - a\chi_n \frac{d}{d\mu}P_n^0$. Thus, there is a critical amplitude a for developing the instability, due to the sphere rotation.

Case 2. (Baines, 1976). Any mode (18) of LP flow (13) of the degree $n = 1$ or $n = 2$ is linearly stable.

Case 3. (Skiba and Adem, 1998). Let $n \geq 3$. Any mode (18) of LP flow (13) may be unstable only if zonal wavenumber m of the mode satisfies the condition $0 < |m| < n$.

Case 4. (Skiba, 2000). Let $\tilde{\psi}$ be a LP flow (13) or RH wave (16). Then the amplitude Ψ of each unstable or decaying mode (10) must satisfy the condition

$$\chi_\Psi = \frac{\eta(\Psi)}{K(\Psi)} = \chi_n = n(n+1) \tag{19}$$

where χ_Ψ is the square of Fjörtoft's (1953) average spectral number of the mode amplitude Ψ, and $K(\Psi) = \frac{1}{2}\|\nabla\Psi\|^2$ and $\eta(\Psi) = \frac{1}{2}\|\Delta\Psi\|^2$ are the total kinetic energy and enstrophy of Ψ.

Case 5. (Skiba and Strelkov, 2000a,b). Let $\tilde{\psi}$ be a monopole or dipole modon by Verkley (1990) with uniform absolute vorticity in the inner region S_i. Then the amplitude Ψ of each unstable or decaying mode (10) must satisfy the condition

$$\chi_\Psi = \chi_\sigma = \sigma(\sigma + 1) \tag{20}$$

where σ is the degree of the modon in the outer region. Moreover, the vorticity of each unstable mode must be zero in S_i.

Also we have used the equation

$$\frac{dK}{dt} = -\int_S \sqrt{1-\mu^2}\frac{d}{d\mu}\tilde{u}(\mu) \cdot (uv)ds - \int_S \frac{\mu}{\sqrt{1-\mu^2}}\tilde{u} \cdot (uv)ds \tag{21}$$

that describes the evolution of the total energy $K(\psi) = \frac{1}{2}\|\nabla\psi\|^2$ of an infinitesimal perturbation $\psi(\lambda, \mu, t)$ to a zonal flow $\tilde{\psi}(\mu)$ on the sphere S where

$$\tilde{u} = -\sqrt{1-\mu^2}\frac{\partial\tilde{\psi}}{\partial\mu}, u = -\sqrt{1-\mu^2}\frac{\partial\psi}{\partial\mu}, v = \frac{1}{\sqrt{1-\mu^2}}\frac{\partial\psi}{\partial\lambda} \tag{22}$$

are the velocity components of the basic flow and the perturbation, respectively. This equation is useful in the analysis of the spatial structure of local unstable perturbations on the sphere. Usually, important localized perturbations are in the regions related with distinguishing features of the basic flow. Unlike the perturbation energy equation on the β-plane (Pedlosky, 1979), its spherical analogue (21) contains one more integral of the product of the basic velocity $\tilde{u}$ with uv. Whereas the first integral dominates principally at the sides of the basic jets located in the tropics and mid-latitudes (where both $\frac{d}{d\mu}\tilde{u}(\mu)$ and $\sqrt{1-\mu^2}$ are large), the second integral can be significant in the central parts of strong jets (where velocity $\tilde{u}$ is large), and especially when such jets are located in the polar regions (where $\frac{\mu}{\sqrt{1-\mu^2}}$ is also large).

By (21), the sign of $\frac{d}{dt}K$ depends on the signs of the products $\frac{d}{d\mu}\tilde{u}(\mu)\cdot(uv)$ and $\tilde{u}\cdot(uv)$ in various regions of the sphere. In the case when the first integral is dominant, one can say that the growth (or decrease) of the perturbation energy takes place in the regions where the inclination of the isolines of the perturbation streamfunction ψ is opposite to (coincides with) the inclination of the profile of the basic velocity $\tilde{u}(\mu)$, that is, in the regions where product $\frac{d}{d\mu}\ \tilde{u}(\mu)\cdot uv$ is positive (negative) (Pedlosky, 1979).

4. Numerical experiments

We now describe the numerical normal mode stability study results obtained with our algorithm for the LP flows, zonal RH waves and monopole modons. In all the experiments, the spectral problem (12) is solved with the QR-factorization (Wilkinson, 1965). The skill of the normal mode stability study algorithm has been checked with the following facts:

a) the eigenvalues must usually appear in groups of 4 eigenvalues located symmetrically about the origin on the complex plane (Arnold, 1965);

b) any LP flow of degree one or two must be stable and have only neutral modes (Case 2);

c) for a basic LP flow of degree n, the zonal wavenumber m of each unstable mode must belong to interval $0 < |m| < n$ (Case 3);

d) any LP flow of degree n as well as any zonal RH wave may have unstable mode only if Rayleigh-Kuo condition is satisfied (Case 1);

e) if degree n of the LP flow is odd then all the modes are divided into two groups that contain just symmetric or antisymmetric (with respect to the equator) modes (Skiba, 1989; Skiba and Adem, 1998);

f) spectral number of the amplitude of each unstable mode must be equal to a special number (Cases 4 and 5);

g) the amplitude of each unstable mode must be orthogonal to the basic flow in the energy inner product (Skiba, 2000);

h) the growth rate of the most unstable mode must be limited from above by the theoretical estimations (Skiba, 2000);

i) for a modon by Verkley (1990), the vorticity of each unstable mode must be zero in the inner modon region S_i;

j) the generation and dissipation of the perturbation energy of a zonal flow should be explained with formula (21).

The requirements a) and h) have been satisfied in al! the experiments. By the requirement g), the amplitude of each unstable mode must be orthogonal to the basic flow in the energy inner product: $\left\langle \tilde{\psi}, \Delta\psi \right\rangle = 0$. This property is always obeyed for the zonal flows, since all the calculated modes have the form (18), that is, have a harmonic structure in the λ-direction:

$$\left\langle \tilde{\psi}, \Delta\psi \right\rangle = \left\langle \Delta\tilde{\psi}, \psi \right\rangle = e^{\omega t} \int_{-1}^{1} \Delta\tilde{\psi}(\mu)\psi^*(\mu) \left\{ \int_0^{2\pi} e^{-im\lambda} d\lambda \right\} d\mu = 0$$

a) Legendre-polynomial flow (13).

For such flows, the structure of the stability matrix L (see (9)) was analytically analyzed by Skiba and Adem (1998).

1. Let $n = 1$. The stability matrix L is then diagonal with pure imaginary diagonal elements $L_{\alpha\alpha}$. In this case, the eigenvalues coincide with the diagonal elements, and hence, $\omega_r = 0$ for any eigenvalue. Therefore all the modes are neutral regardless of the amplitude a of the LP flow. The amplitude of each mode coincides with the corresponding spherical harmonics $Y_\alpha(\lambda, \mu)$, and hence is symmetric or antisymmetric about equator depending on whether $n_\alpha - m_\alpha$ is even or odd.

2. Let $n = 2$. Then stability matrix L is tridiagonal, and the calculations give $\omega_r = 0$ for all eigenvalues, that is, all the modes are neutral. Thus, any LP flow of the degree one or two is exponentially stable in full agreement with Case 2.

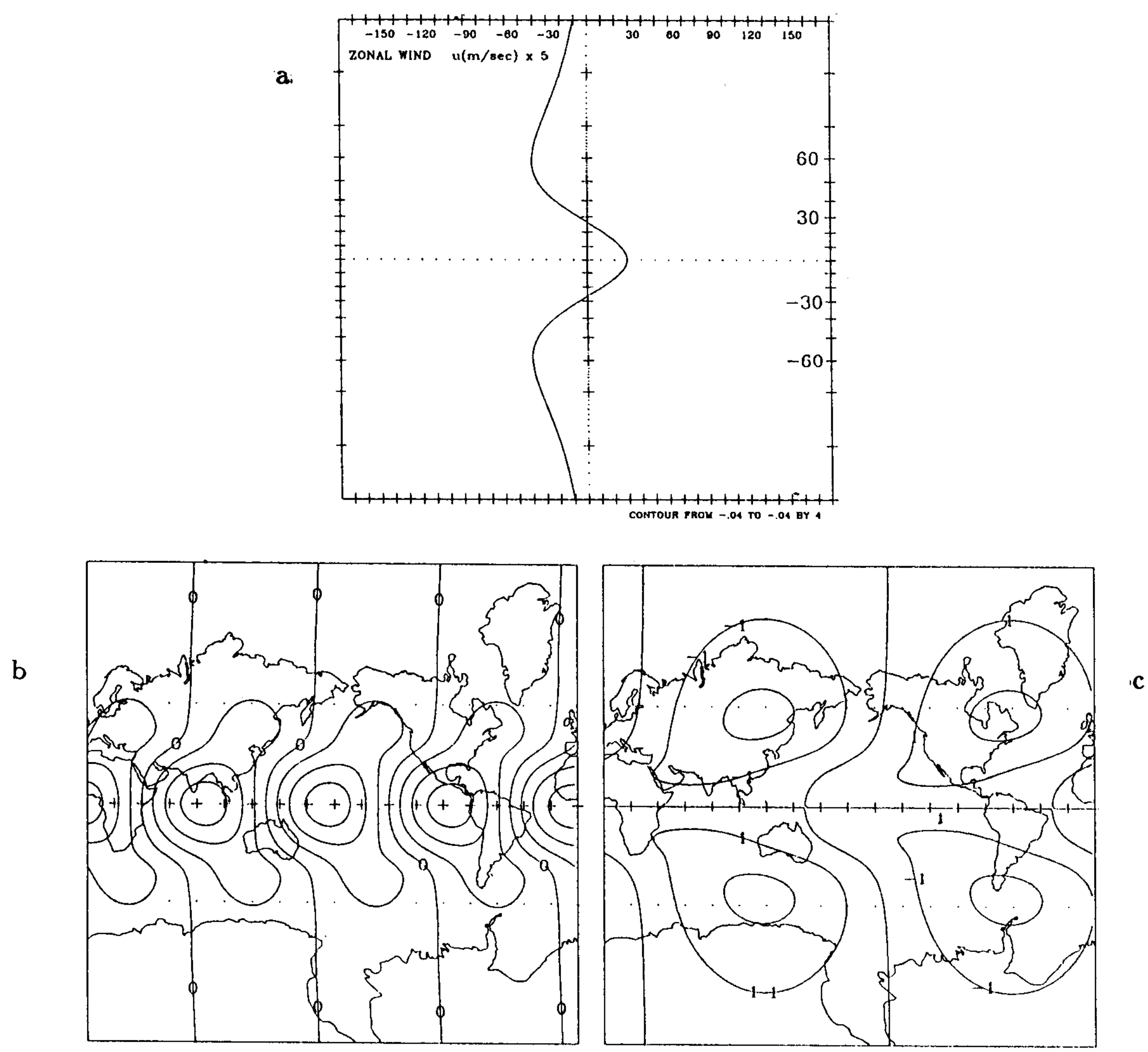


Fig. 1. The profile of the velocity $\widetilde{u}(\mu)$ of the Legendre-polynomial flow (13) with $n = 3$ and $a = 0.08$ (Fig. 1a), and isolines of the amplitudes $\Psi_r(\lambda, \mu)$ of two most unstable modes corresponding to $\omega_r = 0.2073$ (Fig. 1b) and $\omega_r = 0.1576$ (Fig. 1c).

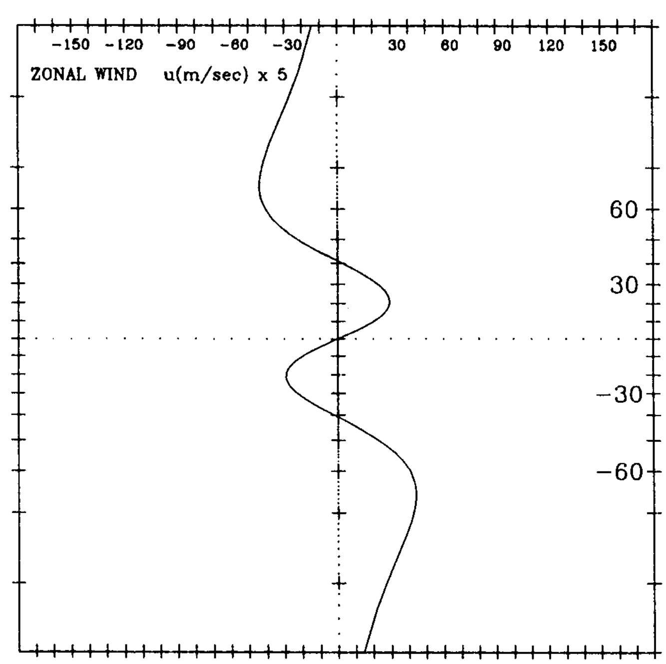
3. Let $n = 3$ and $a = 0.08$. Here n is odd, and in accordance with requirement e), the amplitude of all the modes can be just of the two types: symmetric or antisymmetric around the equator $\mu = 0$. The most unstable mode (18) is symmetric around the equator, has $\omega_r = 0.2073$ and zonal wavenumber $m = 2$, and spectral number χ_Ψ of its amplitude is equal to 11.999 (the corresponding number provided by Case 4 is $\chi_3 = 12$). The second unstable mode has $\omega_r = 0.1576$ and zonal wavenumber $m = 1$, is antisymmetric with respect to equator, and spectral number χ_Ψ of its amplitude is 11.999 as well. The profile of the basic flow velocity $\tilde{u}$ is shown in Figure 1a, while the real parts Ψ_r of the amplitudes of the first and second most unstable modes are presented in Figure 1b and Figure 1c, respectively. For convenience in using formula (21), the figures are marked with the symbols "+ + +" and "· · ·" at the positions of the basic flow jets with positive and negative velocities $\tilde{u}$, respectively. Indeed, the profile of $\tilde{u}$ has the same inclination inside each region (channel) between two neighboring lines marked with different symbols, and this inclination is changed in passing from one region to another. It should be noted that for all the modes, nonzero amplitude values are captured in some latitudinal bands (Tung, 1981). An analysis of the fields shows that both integrals of the formula (21) contribute to the instability for the most unstable mode, while the first integral is dominant in generating the instability for the second mode. It should also be noted that values of uv are much smaller in the second case. In the experiments carried out, the magnitude $|\omega_r|$ increases with the basic flow amplitude a. This fact is also in full accordance with the theoretical estimates (requirement h). We use this requirements to control the growth rate of unstable modes. The main parameters of the two unstable modes (zonal wavenumber m, the real part ω_r and imaginary part ω_i of the eigenvalue, spectral number χ_Ψ, e-folding time τ_e, and period T) are given in Table 1.

Table 1. The two most unstable modes of LP flow (13), $n = 3$, $a = 0.08$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	2	0.2073	-0.7334	11.9999	0.76	1.36
2	1	0.1576	0.0670	11.9999	1.00	14.90

4. Let $n = 4$ and $a = 0.06$. The profile of the basic flow velocity $\tilde{u}$ is shown in Figure 2a, while isolines of the real parts $\Psi_r(\lambda, \mu)$ of the amplitudes of the four most unstable modes are presented in Figures 2b-e. Since n is even, the amplitude of all the modes are asymmetric around the equator. Moreover, the mode perturbations are basically located in one of the two hemispheres. Spectral numbers χ_Ψ of their amplitudes

Fig. 2 a.



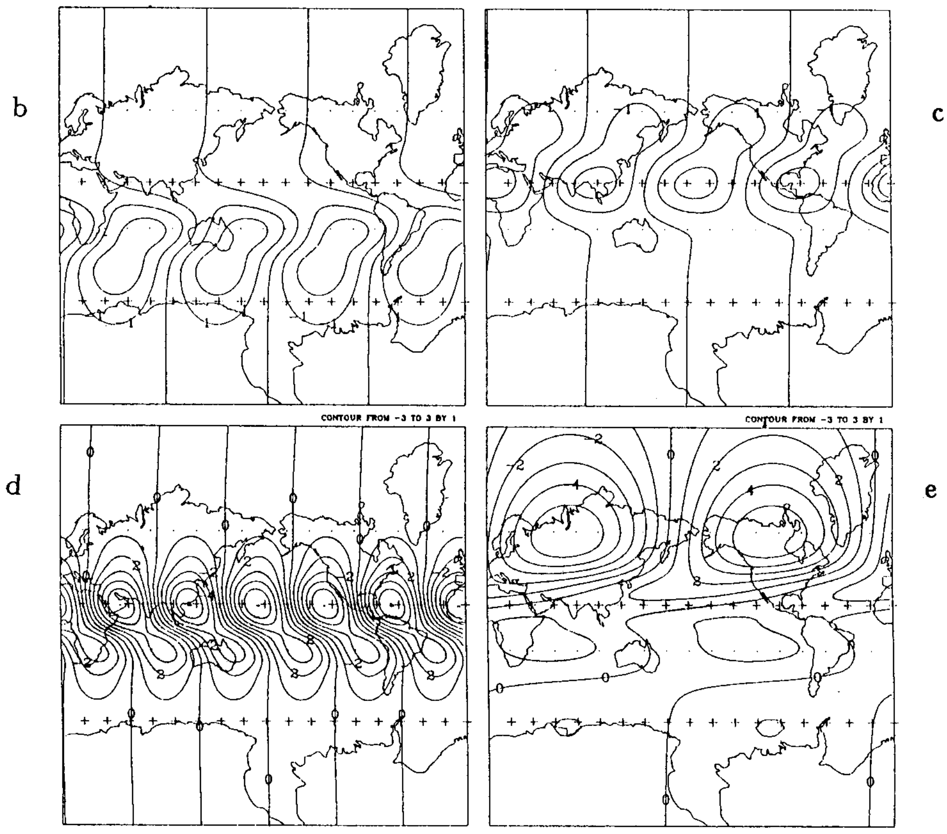


Fig. 2. The profile of the velocity $\widetilde{u}(\mu)$ of the Legendre-polynomial flow (13) with $n = 4$ and $a = 0.06$ (Fig. 2a), and isolines of the amplitudes $\Psi_r(\lambda, \mu)$ of four most unstable modes corresponding to $\omega_r = 0.2604$ (Fig. 2b), $\omega_r = 0.2373$ (Fig. 2c), $\omega_r = 0.1924$ (Fig. 2d), and $\omega_r = 0.1485$ (Fig. 2e).

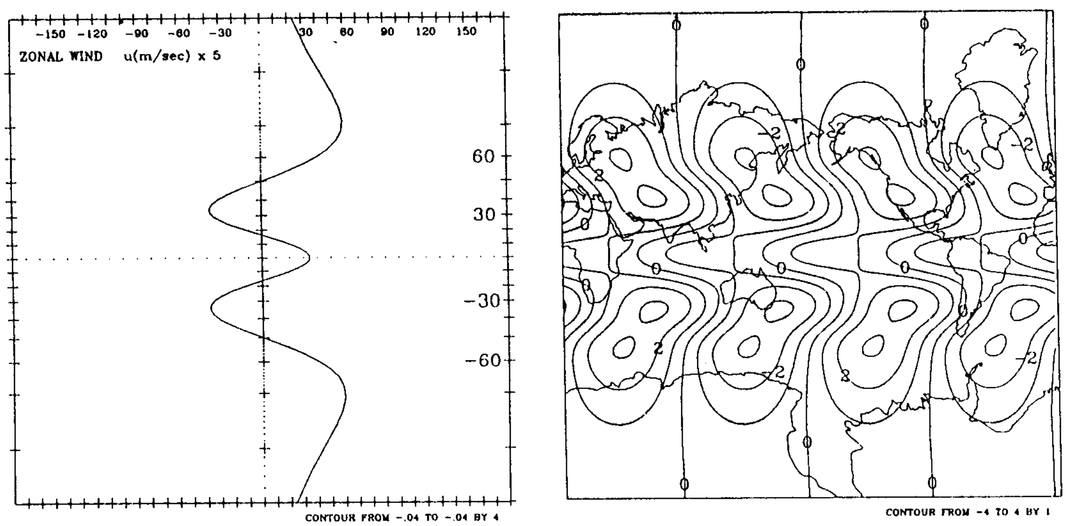
almost coincide with the number $\chi_4 = 20$ provided by Case 4. An analysis of the fields shows that both integrals of formula (21) contribute to the instability for the first three most unstable modes (Figures 2b-d), while the contribution of the first integral in generating the instability for the fourth mode is weaker (Figure 2e). The main parameters of these modes are given in Table 2. It is seen that the requirements c), f) - h) are fulfilled.

Table 2. The four most unstable modes of LP flow (13), $n = 4$, $a = 0.06$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	2	0.2604	-0.1099	19.9999	0.61	9.09
2	2	0.2373	0.6876	19.9999	0.67	1.45
3	3	0.1924	0.4080	19.9999	0.82	2.45
4	1	0.1485	-0.0578	20.0000	1.07	17.28

5. Let $n = 5$ and $a = -0.06$. The profile of the basic flow velocity $\widetilde{u}$ is shown in Figure 3a, while isolines of the real part $\Psi_r(\lambda, \mu)$ of the most unstable mode amplitude are presented in Figure 3b. The zonal wavenumber of the mode is $m = 2$, and mode amplitude is symmetric around the equator (n is odd). The perturbations are located in a neighborhood of the two subtropical jets, and both the integrals of formula (21) contribute to the mode instability with the dominant role of the first one. The main parameters of the seven most unstable modes are given in Table 3. It is seen that the requirements c), f) - h) are fulfilled, and the dominant zonal wavenumbers of the modes are $m = 2$ and $m = 3$.

a



b

Fig. 3. The profile of the velocity $\widetilde{u}(\mu)$ of the Legendre-polynomial flow (13) with $n = 5$ and $a = -0.06$ (Fig. 3a), and isolines of the amplitude $\Psi_r(\lambda, \mu)$ of the most unstable mode corresponding to $\omega_r = 0.4922$ (Fig. 3b).

Table 3. The seven most unstable modes of LP flow (13), $n = 5$, $a = -0.06$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	2	0.4922	0.0592	29.9999	0.32	16.87
2	2	0.4688	0.1809	30.0000	0.33	5.52
3	4	0.3630	0.4042	29.9999	0.43	2.47
4	3	0.3222	0.0110	30.0000	0.49	90.82
5	3	0.2745	0.3886	29.9999	0.57	2.57
6	2	0.1662	-0.6740	29.9999	0.95	1.48
7	1	0.1421	0.1088	29.9999	1.11	9.18

6. Let $n = 6$ and $a = 0.03$. The main parameters of the seven most unstable modes are given in Table 4. Unstable perturbations are located in the Northern Hemisphere (modes 2, 3 and 4), in the Southern Hemisphere (modes 1, 6 and 7), and in the tropics (mode 5). It is seen from Table 4 that the requirements c), f) - h) are fulfilled.

Table 4. The seven most unstable modes of LP flow (13), $n = 6$, $a = 0.03$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	2	0.3471	0.1376	41.9999	0.45	7.26
2	2	0.3435	0.5340	41.9999	0.46	1.87
3	3	0.2700	0.3556	41.9999	0.58	2.81
4	4	0.2463	0.2251	42.0000	0.64	4.44
5	5	0.2137	-0.3101	42.0000	0.74	3.22
6	4	0.1994	0.3646	42.0000	0.79	2.74
7	3	0.1798	0.0786	42.0000	0.88	12.71

7. Let $n = 7$ and $a = 0.01$. The main parameters of the twelve most unstable modes are given in Table 5. The majority of the modes (modes 1, 2, 6, 7 and 8-12) are symmetric around equator (requirement e) and located in the middle latitudes.The requirements c), f) - h) are fulfilled, too.

Table 5: The twelve most unstable modes of LP flow (13), $n = 7$, $a = 0.01$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	2	0.1245	-0.3719	56.0000	1.27	2.68
2	2	0.1181	0.4005	56.0000	1.34	2.49
3	4	0.1153	0.1564	55.9999	1.37	6.39
4	5	0.1115	-0.2715	55.9999	1.42	3.68
5	6	0.1112	-0.2732	56.0000	1.43	3.65
6	4	0.1027	0.1891	55.9999	1.54	5.28
7	3	0.0828	0.3586	55.9999	1.91	2.78
8	5	0.0786	-0.1774	56.0000	2.02	5.63
9	1	0.0755	0.0630	56.0000	2.10	15.86
10	3	0.0739	0.0983	56.0000	2.15	10.16
11	1	0.0727	0.0635	56.0000	2.18	15.73
12	2	0.0645	0.0271	56.0000	2.46	36.81

b) Zonal Rossby-Haurwitz waves (14)

1. Let $n = 5$ and $a = 0.0052$. The profile of the basic flow velocity $\widetilde{u}$ is symmetric around equator (Figure 4a) and has a jet in the middle latitudes. There are just four different unstable modes, and the first of them is quasi-stationary (Table 6). Spectral numbers χ_Ψ of the amplitudes of these modes coincide with the theoretical number $\chi_5 = 30$ provided by Case 4. The real part of mode (18) can also be written as

$$\psi_r(\lambda, \mu, t) = |\Psi(\mu)|\, e^{\omega_r t} \cos m \left(\lambda + \frac{\kappa(\mu)}{m} + \frac{\omega_i}{m} t \right) \tag{23}$$

or

$$\psi_r(\lambda, \mu, t) = e^{\omega_r t} \left[\Psi_r(\lambda, \mu) \cos \omega_i t - \Psi_i(\lambda, \mu) \sin \omega_i t \right]$$

where $\frac{\omega_r}{m}$ is the phase velocity, and the initial phase $\frac{\kappa(\mu)}{m}$ depends on μ. For positive (or negative) $\frac{\omega_r}{m}$ the mode moves to the west (east). Isolines of the real part (23) of the most unstable mode are given in Figures 4b-d at the moments $t = 0$, $t = T/8$ and $t = T/4$, respectively. The mode moves very slowly to the west. Spatial variations of the field (23) with time are due to the dependence of the initial phase $\frac{\kappa(\mu)}{m}$ on μ. The amplitude of the most unstable mode is antisymmetric about the equator (n is odd). An analysis shows that both integrals of formula (21) contribute to the mode instability. The requirements f) - h) are fulfilled.

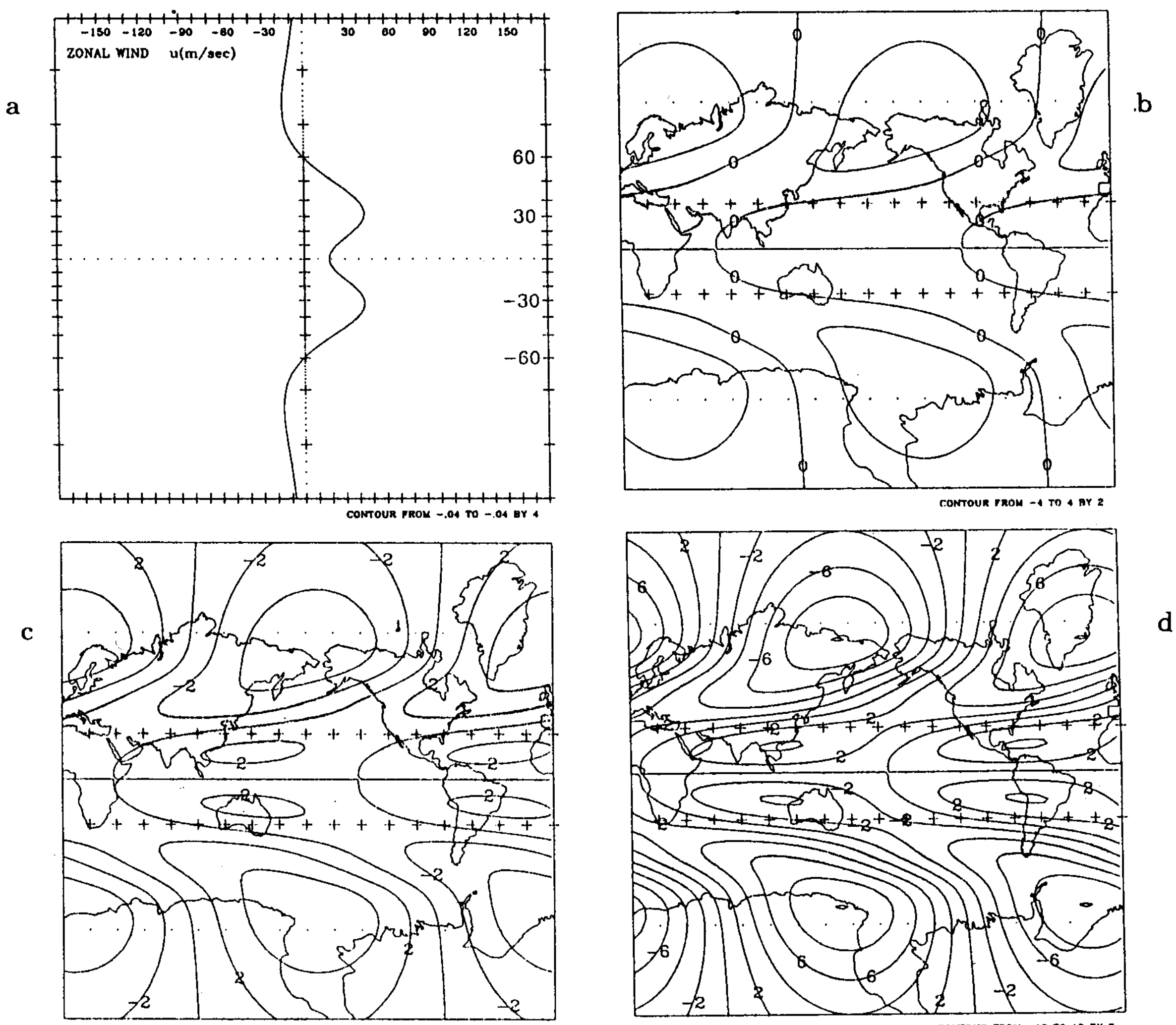


Fig. 4. The profile of the velocity $\widetilde{u}(\mu)$ of the zonal Rossby-Haurwitz wave (14) with $n = 5$ and $a = 0.0052$ (Fig. 4a), and isolines of $\Psi_r(\lambda, \mu, t)$ for the most unstable mode ($\omega_r = 0.0268$) at the moments $t = 0$ (Fig. 4b), $t = T/8$ (Fig. 4c) and $t = T/4$ (Fig. 4d).

Table 6. The four most unstable modes of RH wave (14), $n = 5$, $a = 0.0052$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	1	0.0268	0.0064	30.0000	5.92	155.15
2	1	0.0151	0.0287	30.0000	10.52	34.74
3	3	0.0142	-0.1255	29.9999	11.16	7.97
4	2	0.0041	-0.2510	30.0000	38.20	3.98

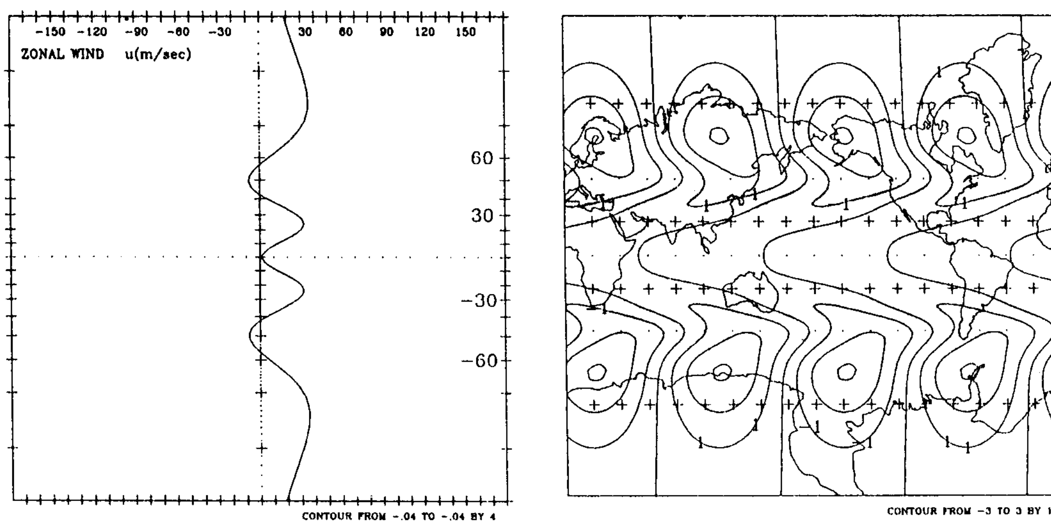
2. Let $n = 6$ and $a = 0.002$. There are just three different unstable modes, besides, the first two of them are quasi-stationary again (see Table 7). Spectral numbers χ_Ψ of the amplitudes of these modes coincide with the number $\chi_6 = 42$ provided by Case 4.

Table 7. The three most unstable modes of RH wave (14), $n = 6$, $a = 0.002$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	1	0.0103	0.0128	42.000	15.31	77.68
2	2	0.0085	0.0303	42.000	18.51	32.90
3	2	0.0014	0.1031	42.000	107.15	9.69

3. Let $n = 7$ and $a = -0.004$. The profile of the basic flow velocity $\tilde{u}$ is symmetric about equator (Figure 5a) and has four jets. The main parameters of the eight most unstable modes are given in Table 8. Almost all of them are quasi-stationary. Spectral numbers χ_Ψ of the amplitudes of these modes are practically coincide with the number $\chi_7 = 56$ provided by Case 4. Isolines of the amplitude $\Psi_r(\lambda, \mu)$ of the most unstable mode corresponding to $\omega_r = 0.0515$ are presented in Figure 5b.

a



b

Fig. 5. The profile of the velocity $\widetilde{u}(\mu)$ of the zonal Rossby-Haurwitz wave (14) with $n = 7$ and $a = -0.004$ (Fig. 5a), and isolines of the amplitude $\Psi_r(\lambda, \mu)$ of the most unstable mode corresponding to $\omega_r = 0.0515$ (Fig. 5b).

Table 8. The eight most unstable modes of RH wave (14), $n = 7$, $a = -0.004$

Modes	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
1	2	0.0515	0.0099	56.000	3.08	100.65
2	2	0.0501	-0.0067	56.000	3.17	148.80
3	3	0.0370	0.0044	56.000	4.29	223.45
4	3	0.0362	0.0035	56.000	4.39	279.75
5	4	0.0266	-0.0093	56.000	5.96	106.53
6	5	0.0157	-0.0554	56.000	10.12	18.03
7	1	0.0128	-0.0011	55.999	12.36	867.33
8	1	0.0081	0.0089	55.999	19.58	111.73

c) Zonal flow in the form of a monopole modon (17)

As one more test we now consider the zonal flow that has the form of the monopole modon (17) with uniform absolute vorticity in the inner region S_i (Verkley, 1990). The main parameters of the modon are

$\sigma = 8.06$ and $\phi_a = 65.277$. The profile of the modon velocity $\widetilde{u}$ (Fig.6a) has three jets in the outer modon region S_o. The most unstable mode has been calculated with three different truncation numbers: T_{21},T_{31} and T_{42}. Isolines of the real part $\Psi_r(\lambda, \mu)$ of the amplitude of this mode are presented in Figures 6b, 6c and 6d, respectively. The basic parameters of the mode are given in Table 9. In Fig. 6b-d, nonzero perturbations are in a latitudinal belt located in the outer modon region S_o, as it must be according to Case 5. In this belt, the inclination of the principle axes of the vortices is opposite to that of the modon velocity profile, the result being in full accordance with the first integral of the formula (21). The second integral in (21) contributes in the energy growth much less than the first one, and basically in the vicinity of the modon jet center that contains important perturbations. The polar regions contain no perturbations and make no impact on the energy behavior.

a b c d

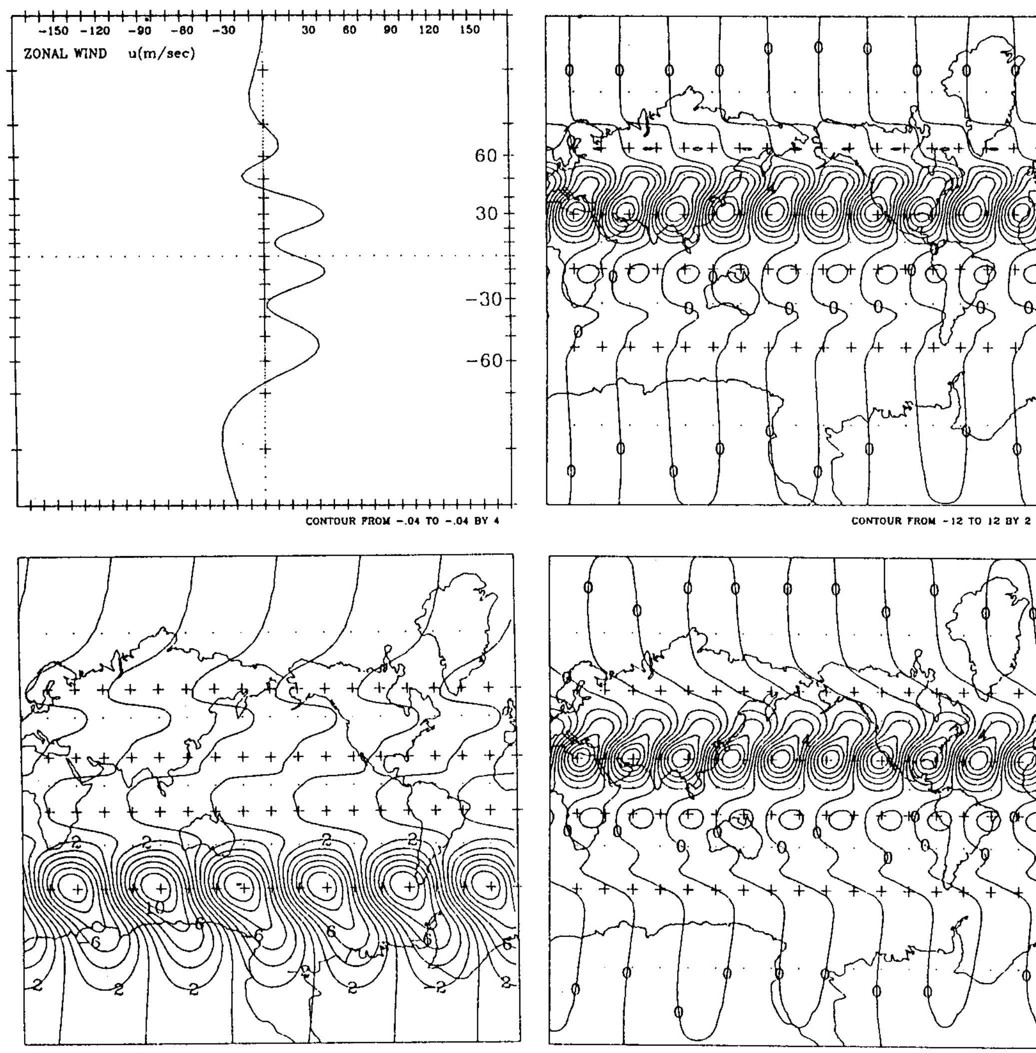


Fig. 6. The profile of the velocity $\widetilde{u}(\mu)$ of the monopole modon (17) with $\sigma = 8.06$ and $\phi_a = 65.277$ (Fig. 6a), and isolines of the amplitudes $\Psi_r(\lambda, \mu)$ of the most unstable mode at resolutions T_{21} (Fig. 6b), T_{31} (Fig. 6c) and T_{42} (Fig. 6d).

Table 9. The most unstable mode of modon (17) for T_{21}, T_{31} and T_{42}, $\sigma = 8.06$, $\phi_a = 65.277$

Truncation number	Mode	m	ω_r	ω_i	χ_Ψ	τ_e days	Period T days
T_{21}	1	5	0.0524	0.0177	80.72	3.03	56.34
T_{31}	1	3	0.0627	0.0373	70.56	2.53	26.74
T_{42}	1	5	0.0569	0.0457	74.60	2.79	21.87

It should be noted however that to achive a good agreement of the numerical and theoretical results the spectral resolution up to T_{42} is still insufficient. Indeed, although Table 9 shows a convergence of the numerical results as truncation numbers M and N of the Fourier series (4) increase, this convergence is rather slow. For the most unstable mode, for example, one can still see a difference between the theoretical value $\chi_\sigma = \sigma(\sigma+1) = 73.02$ of the spectral number χ_Ψ (according to the Case 5) and its numerical value which is equal to 80.72 at resolution T_{21}, 70.56 at resolution T_{31}, and 74.60 at resolution T_{42}. For the second unstable mode, the corresponding numerical values are 78.81 at T_{21}, 73.95 at T_{31}, and 73.11 at T_{42}, that is, they are closer to the theoretical value. Besides, as it is seen from Figure 6b-d, the most unstable mode is of the same structure at resolutions T_{21} and T_{42}, but its structure is changed at resolution T_{31}. This situation occurs when the modon has a few (at least two) most unstable modes with approximately equal real parts ω_r. Dikii (1976) showed that unstable modes of a zonal flow always correspond to the isolated eigenvalues. However, if the distance between two isolated eigenvalues with maximum real parts ω_r is very small then a high resolution is required to approximate well a concrete eigenvalue and the corresponding mode. Thus, the change of the structure of the most unstable mode is due to insufficient resolution and quite large truncation errors (Simmons *et al.* 1983, Skiba, 1998). And although spectral numbers χ_Ψ for the first two most unstable modes calculated tend to the theoretical value $\chi_\sigma = 73.02$ as truncation numbers M and N of the Fourier series (4) increase, the convergence is slow, and a higher resolution is required to get more precise results. This phenomenon is explained by the fact that the μ-derivative of the modon vorticity is discontinuous at the boundary $\mu = \mu_a$ separating the regions S_i and S_o on the sphere (Verkley, 1990; Neven, 1992), that is, the modon is not so smooth as LP flows or RH waves. As a result, the series of the spherical harmonics for the modon and its perturbation converge (in Sobolev's norms) slower than the corresponding fields for the LP flows and RH waves (Topuria, 1987; Skiba, 1989, 1994, 1997). Besides, due to the Gibbs phenomenon (Davis, 1963), the convergence of Fourier series is more rapid inside the regions S_i and S_o than in a vicinity of the boundary $\mu = \mu_a$, so that, no matter how large M and N may be, at some points of S the finite sums (4) are very nearly the same as the functions $\widetilde{\psi}$, ψ and ξ, but at other points these sums differ substantially from $\widetilde{\psi}$, ψ and ξ. Thus, truncation numbers M and N considerably greater than 42 must be used in (4) to reduce numerical errors. The importance of using a high resolution for the solution of the spectral problem (12) was recently shown theoretically by Skiba (1998) and numerically by Neven (2000a,b) who has used the resolution T_{341} and the power method in the normal mode stability study of modons on a sphere.

5. Conclusions

In the present work, we have tested a numerical algorithm developed for the normal mode (exponential) instability of arbitrary steady flow of an ideal nondivergent fluid on a rotating sphere. Only zonal solutions to the BVE (Legendre polynomial flows, steady zonal Rossby-Haurwitz waves and monopole modons) have been examined here. On trials, the skill of the normal mode stability study algorithm has been checked by comparing the numerical stability results with the theoretical requirements a)-j) formulated in the beginning of Section 4. For the LP flows and zonal RH waves the numerical stability results have a very good accuracy. All the theoretical requirements a)-j) have been satisfied. In particular, in all the figures representing the unstable mode streamfunction ψ, the inclination of ψ-isolines is opposite to that of the basic velocity profile. According to the first integral of the perturbation energy equation (21), this leads to the perturbation energy growth (requirement j)). As to the requirements c), we note that in almost 50% of the experiments carried out, the most unstable mode has zonal wavenumber $m = 2$. The next two popular zonal wavenumbers are $m = 3$ and $m = 1$. This is also in good agreement with the results previously obtained by Baines (1976).

The numerical results on the monopole modon instability are also satisfactory. The most part of the requirements a)-j) has been satisfied. However, it should be noted that the modon is not so smooth as LP flows and RH waves (the derivative of the modon vorticity is discontinuous on a sphere), and as a result, the series of the spherical harmonics for the modon and its perturbation converge slowly as the truncation numbers M and N increase (see Table 9). We have found two manifestations of the slow convergence: 1) the structure of the most unstable mode is changed at resolution T_{31} being the same at resolutions T_{21} and T_{42}; and 2) in comparison with the LP flows and RH waves, spectral number $\chi_\Psi = \eta(\Psi)/K(\Psi)$ of the amplitude of an unstable mode converges rather slowly to its theoretical value. Thus, to get the modon stability results with a high degree of accuracy, a resolution considerably higher than T_{42} is required.

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